Spurious momentum mismatch introduced by an approximate model of acousto-optic interactions

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In acousto-optic interactions, the concept of momentum mismatch introduced by some books is misleading and nonexistent because the mismatch itself is caused by the approximate model used to explain the interactions. The so-called mismatch is very small and should not be handled with an approximate model. We use an exact model to satisfy Bragg condition, conservation of energy, and conservation of momentum. The difference between the exact model and the approximate model is actually what causes the mismatch.
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In acousto-optic interactions, the concept of momentum mismatch introduced by some books is surprising.[1,2] Examples are generally given to support the concept, but an unclosed momentum diagram is often the evidence that conservation of momentum is violated. At the same time, those who do not accept the idea of momentum mismatch have not offered any reason to explain the unclosed diagram. Because of this controversy, the description of acousto-optic interactions given in some textbooks is incomplete.

The problem arises because an approximate model, which considers the angles of incidence and diffraction to be equal, is used to describe the interactions. We will show that strictly speaking these angles are not in fact equal. The resulting so-called mismatch is very small and cannot be handled with an approximate model.

We resolve the momentum mismatch problem by using an exact model that satisfies simultaneously the Bragg condition, conservation of energy, and conservation of momentum. The tiny difference between the exact model and the approximate model is the source of the ''momentum mismatch.''

The diffraction of light by sound waves can be described as an interaction with three particles, the incident photon, the diffracted photon, and the acoustic phonon, which have to satisfy the Bragg diffraction condition. A light wave of frequency ω and propagation vector $\mathbf{k}$ can be considered as a stream of photons with momentum $\hbar\mathbf{k}$ and energy $\hbar\omega$. Likewise, the sound wave of frequency Ω and propagation vector $\mathbf{K}$ can be thought of as phonons with momentum $\hbar\mathbf{K}$ and energy $\hbar\Omega$. The widely used three-particle momentum diagram is shown in Fig. 1. In this model, both the incident angle and the diffracted angle are taken to be the same as the Bragg angle.

Strictly speaking, however, the two angles are slightly different. In relativistic optics, when a plane mirror moves in a direction perpendicular to its surface, the angle of reflection changes. In Fig. 2, with the mirror moving toward the light source, the reflection angle α_2 becomes less than the incident angle α_1.

In acousto-optics, the sound wave in the crystal induces a variation of the dielectric constant that causes partial reflection of the incident light into the direction of the diffracted light. Thus the sound wave is analogous to a moving diffraction grating with rulings separated by a distance equal to the acoustic wavelength Λ. We can find the relation between the direction of the incident wave ($\mathbf{k}_1$) and the direction of the diffracted wave ($\mathbf{k}_2$) by the following equations,[3,4] where the angles α_1, α_2 are as shown in Fig. 2:

$$\frac{\tan(\alpha_1/2)}{\tan(\alpha_2/2)}=\frac{(c/n)+v}{(c/n)-v}. \tag{1}$$

Equation (1) can be rewritten as

$$\frac{\sin[(\alpha_1+\alpha_2)/2]}{\sin[(\alpha_1-\alpha_2)/2]}=\frac{c}{nv}. \tag{2}$$

Here, c is the speed of light in vacuum, n is the refractive index, and v is the velocity of the moving surface, in this case the acoustic wave front traveling in the crystal. For simplicity, only the isotropic case is discussed here, but the same treatment applies in anisotropic materials.

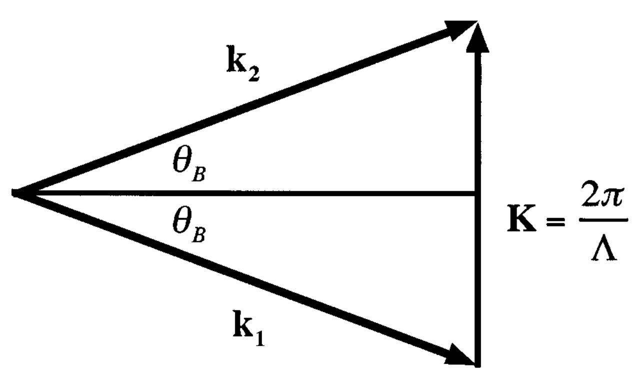

FIG. 1. Traditional momentum diagram. Λ: The wavelength of the acoustic wave. θ_B: The Bragg angle.

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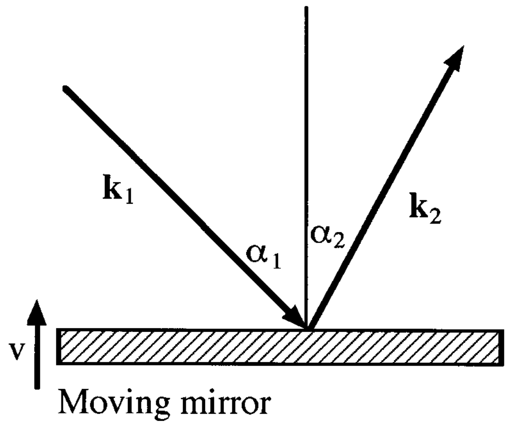

FIG. 2. With the mirror moving toward the source, the angle of reflection becomes less, $\alpha_1 > \alpha_2$. (Adapted from Ref. 4.)

Next, in order to satisfy the Bragg condition, we have to consider the case for constructive interference in the direction of $\mathbf{k}_2$, as shown in Fig. 3.

For constructive interference

$$k_1 \Lambda \cos \alpha_1 + k_2 \Lambda \cos \alpha_2 = 2 m \pi, \tag{3}$$

where m is an integer. We can let $m=1$ and rearrange the equation as

$$k_1 \cos \alpha_1 + k_2 \cos \alpha_2 = \frac{2\pi}{\Lambda}. \tag{4}$$

Now we can construct an exact momentum diagram in Fig. 4 and check the conservations of energy and momentum at the same time.

From Fig. 4, the conservation of momentum is exactly satisfied by $\hbar \mathbf{k}_2 - \hbar \mathbf{k}_1 = \hbar \mathbf{K}$ and the conservation of energy requires $\omega_2 - \omega_1 = \Omega$, which we will now prove. Since ω_2, ω_1, and Ω are the frequencies of the diffracted wave, the incident wave, and the sound wave, respectively, we have to prove that

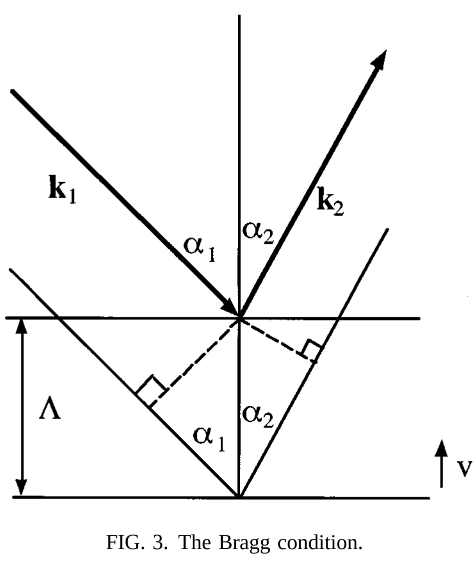

FIG. 3. The Bragg condition.

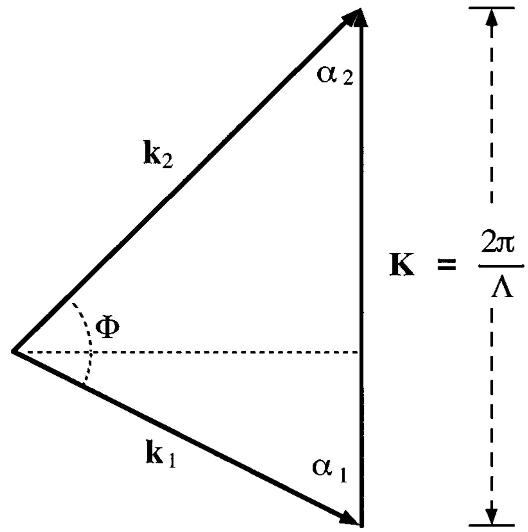

FIG. 4. The exact model.

$$k_2 - k_1 = \frac{n\Omega}{c},$$

$$\left(k_2 = |\mathbf{k}_2| = \frac{n\omega_2}{c}, \quad k_1 = \frac{n\omega_1}{c}, \quad K = \frac{\Omega}{v} \right). \tag{5}$$

The proof is as follows. By the law of sines,

$$\frac{k_1}{\sin \alpha_2} = \frac{k_2}{\sin \alpha_1} = \frac{2\pi/\Lambda}{\sin \Phi} = \frac{k_2 - k_1}{\sin \alpha_1 - \sin \alpha_2}, \tag{6}$$

where $\Phi = \pi - (\alpha_1 + \alpha_2)$. Solving for $k_2 - k_1$ from Eq. (6)

$$k_2 - k_1 = \left(\frac{2\pi}{\Lambda} \right) \frac{\sin \alpha_1 - \sin \alpha_2}{\sin \Phi}$$

$$= \left(\frac{2\pi}{\Lambda} \right) \frac{\sin \alpha_1 - \sin \alpha_2}{\sin[\pi - (\alpha_1 + \alpha_2)]},$$

$$k_2 - k_1 = \left(\frac{2\pi}{\Lambda} \right) \frac{2 \sin[(\alpha_1 - \alpha_2)/2] \cos[(\alpha_1 + \alpha_2)/2]}{\sin(\alpha_1 + \alpha_2)},$$

$$k_2 - k_1 = \left(\frac{2\pi}{\Lambda} \right) \frac{2 \sin[(\alpha_1 - \alpha_2)/2] \cos[(\alpha_1 + \alpha_2)/2]}{2 \sin[(\alpha_1 + \alpha_2)/2] \cos[(\alpha_1 + \alpha_2)/2]}.$$

Using Eq. (2),

$$k_2 - k_1 = \left(\frac{2\pi}{\Lambda} \right) \frac{2 \sin[(\alpha_1 - \alpha_2)/2]}{2 \sin[(\alpha_1 + \alpha_2)/2]} = \frac{2\pi}{\Lambda} \frac{nv}{c} = \frac{n\Omega}{c}.$$

But $k_{1,2} = n\omega_{1,2}/c$, so $\omega_2 - \omega_1 = \Omega$. Therefore, the exact model proposed here simultaneously satisfies the Bragg condition, conservation of energy, and conservation of momentum. Previous models for the acousto-optic interaction have all involved approximations. For example, the momentum mismatch model[1,2] takes the angles of incidence and reflection to be equal. Because of the different lengths of their respective **k** vectors, the momentum diagram does not close. If we are interested in the difference between k_1 and k_2, we cannot neglect the difference between the two angles. The approximation of the two angles is what causes the momentum mismatch in further calculations involving $k_2 - k_1$. In practice the acoustic frequency Ω is tiny compared to the optical ω's, so this approximate model serves for most practical purposes, although it is not strictly correct.

Compared to the frequency of the light, the frequency shift is very small but it can be measured accurately in the laboratory because the frequency of light is very high. The slight difference in the angles α_1, α_2, however, would not normally be observed due to the finite diffraction angle. For an acousto-optic diffraction of light of 3×10^{14} Hz and acoustic frequency of 500 MHz in a material such as sapphire (Al_2O_3 $n=1.76$, $v=11\times10^3$ m/s), the angular difference is only 0.0074°. Nevertheless, the difference in the diffraction angle is necessary for the frequency shift in the diffracted beam to exist.

In physical reality, the wave fronts of light and sound are not plane. It is impossible to generate a true single plane wave of light. However, it is well known that any field satisfying Helmholt's equation may be decomposed into plane waves. Considering the angular plane-wave spectrum[5] of waves can help explain the deviation from the propagation direction. From the wave interaction point of view, there exists a one-to-one correspondence between plane waves of sound and plane waves of light, which could also be illustrated quantum mechanically by the vector diagram.

In summary, an exact model for acousto-optic interactions is proposed, which makes no approximations and satisfies the Bragg condition, conservation of energy, and closes the momentum diagram. Unlike other approaches, it is easy to understand and does not have to prove independently the Bragg condition and Doppler-shift formula.

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[1] J. Xu and R. Stroud, *Acousto-Optic Devices: Principles, Design, and Applications* (Wiley, New York, 1992), p. 587.
[2] A. Yariv and P. Yeh, *Optical Waves in Crystals* (Wiley, New York, 1984), p. 380.
[3] W. Rindler, *Introduction to Special Relativity*, 2nd ed. (Oxford University, New York, 1991), p.48.
[4] J. R. Meyer-Arendt, *Introduction to Classical and Modern Optics*, 3rd ed. (Prentice-Hall, Englewood Cliffs, NJ, 1989), pp. 430–431.
[5] A. Korpel, *Acousto-optics* (Marcel Dekker, New York, 1988).