Cohesive Behaviors of Multiagent Systems With Information Flow Constraints

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***Abstract*—Bacteria, bees, and birds often work together in groups to find food. A group of mobile wheeled robots can be designed to coordinate their activities to achieve a goal. Networked cooperative uninhabited air vehicles (UAVs) are being developed for commercial and military applications. In order for such multiagent systems to succeed it is often critical that they can both maintain cohesive behaviors and appropriately respond to environmental stimuli. In this paper, we characterize cohesiveness of discrete-time multiagent systems as a boundedness or stability property of the agents' position trajectories and use a Lyapunov approach to develop conditions under which local agent actions will lead to cohesive group behaviors even in the presence of *i*) an interagent "sensing topology" that constrains information flow, where by "information flow," we mean the sensing of positions and velocities of agents, *ii*) a random but bounded delay and "noise" in sensing other agents' positions and velocities, and *iii*) noise in sensing a resource profile that represents an environmental stimulus and quantifies the goal of the multiagent system. Simulations are used to illustrate the ideas for multivehicle systems and to make connections to synchronization of coupled oscillators.**

***Index Terms*—Multiagent systems, multivehicle systems, stability analysis, swarms, synchronization of coupled oscillators.**

I. Introduction

COOPERATIVE multiagent systems, sometimes called "swarms," have been studied extensively in biology [1], [2], and physics [3], [4] where collective behavior of "self-propelled particles" is studied. Swarms have also been studied in the context of engineering applications, including in collective robotics where there are teams of robots working together by communicating over a communication network [5], [6], in "intelligent vehicle highway systems" [7], [8], and in "formation control" for robots, aircraft, and cooperative control for uninhabited autonomous (air) vehicles [9], [10]. Swarm stability has been studied in a continuous time ordinary differential equation framework in, for instance, [10]–[12]. Early work in swarm stability was done in a discrete-time framework [13], [14]. Subsequent work studied the one-dimensional discrete-time asynchronous case with time delays in [15], [16]. The higher dimensional case, where there is asynchronism, delays, and a fixed "line" communication topology, is considered in [17]. A coordinated control strategy is studied in [18] for a swarm to climb resource profile when its gradient is not readily available. Recently, by using graph theory, some progress has been made in the study of cohesion properties of multiple agents interconnected by a switching topology [19], [20].

In this paper, we continue some of our earlier work by studying stability properties of foraging swarms [12], which has its synchronous discrete-time version in [21]. The main difference with our previous work is that we consider the effects of an interagent "sensing topology" and random but bounded sensing delays in an asynchronous discrete-time framework. The topology and delays both impose information flow constraints on the multiagent system (where by "information flow," we mean the sensing of positions and velocities of agents), which significantly complicate its ability to achieve cohesive and purposeful behavior. While in [12] we exploited a large-scale system stability methodology with scalar Lyapunov functions and M-matrices [22], here we extend some ideas from the theory of numerical methods in distributed computing [23], particularly, the *agreement algorithm* developed there. However, it should be noted that there are significant differences between our model and the agreement algorithm model since: *i*) there are no dynamics in [23] like the second-order dynamics we use for our agents, *ii*) all values possessed by the "processors" in [23] are always bounded by the formulation of the agreement problem, *iii*) there is no receiving (or passing) noise in [23], *iv*) among their interactions there exists no "repulsion" effect among the values possessed by each processor, and *v*) there exists no external effect on the processors analogous to our "resource profile" that models an environmental stimulus. With all these differences, we are able to show that under certain conditions, even with noisy measurements and the group objective of following a resource profile, the group can become cohesive in the sense that interagent distances are uniformly ultimately bounded. Moreover, under some conditions a set of zero interagent distances is exponentially stable. We show this via a general Lyapunov approach like in [24]. Some related work in using consensus methods for vehicular applications includes [25]–[27]. Under the framework of graph theory, the authors in [25] show that there exists an information update strategy for a multiagent system, characterized by a continuous model with time-varying topology, to reach global consensus asymptotically if and only if the communication graph has a spanning tree. In [26], the author studies a model where zero-mass agents have a time-varying communication topology, and provides conditions for such a system to achieve convergence. The authors in [27] investigate the conditions for formation control of a multiunicycle system with a static information flow topology.

The remainder of this paper is organized as follows. In Section II, we introduce a generic model for agents, interactions,

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and the environment. Section III holds the main results on stability and boundedness analysis of cohesion. Section IV holds the results of simulations and applications of the theory, and some concluding remarks are provided in Section V.

Notation

$\Re^n$ denotes the n-dimensional Euclidean space, $\Re^+$ is the nonnegative reals, $\Re^{n\times m}$ denotes the set of all $n \times m$ real matrices, and $\mathcal{Z}^+$ is the nonnegative integers. $\|\cdot\|$ is the Euclidean vector norm. When x is a vector, x_i is the ith component of x. The notation $x > y$ ($x \geq y$), where x and y are vectors of the same dimension, means that $x_i > y_i$ (respectively, $x_i \geq y_i$) for all i. Similarly, if x, y and z are all vectors with the same dimension, $x = \max\{y, z\}$ ($x = \min\{y, z\}$) means its component $x_i = \max\{y_i, z_i\}$ (respectively, $x_i = \min\{y_i, z_i\}$) for all i, and x_i equals to either y_i or z_i when $y_i = z_i$. $|x|$ is the absolute value of x and it is taken componentwise when x is a vector. $\mathbf{1}_n$ denotes the vector $[1, 1, \ldots, 1]^\top \in \Re^n$. $\lceil x \rceil$ denotes the minimum integer that is larger than $x \in \Re^+$.

II. Mathematical Model

A. Agents, Sensing Topology, Interactions, and Environment

Here, we consider a system composed of an interconnection of N ($N \geq 2$) "agents," where the ith agent, $i = 1, \ldots, N$, has point mass dynamics given by

$$\begin{aligned} x^i((k+1)T) &= x^i(kT) + v^i(kT)T \\ v^i((k+1)T) &= v^i(kT) + \frac{1}{M_i}u^i(kT)T \end{aligned} \tag{1}$$

where $x^i \in \Re^n$ is the position, $v^i \in \Re^n$ is the velocity, $M_i > 0$ is the mass, $u^i \in \Re^n$ is the control input, and $T > 0$ is the sampling time. To simplify notation, throughout the paper we replace all "(kT)" with "(k)." Also, we mainly consider the case of $k \in \mathcal{Z}^+$. We assume $x^i(k) = x^i(0)$ for $k < 0$ (and thus, $v^i(k) = 0$ for $k < 0$), $i = 1, \ldots, N$. A double integrator model is used here for each agent since for the study of the coordination level of multiagent systems it is reasonable to assume that a low-level (inner-loop) controller would compensate for fast and nonlinear dynamics.

Let $\mathcal{N} = \{1, \ldots, N\}$ be a set of indices that label the agents. Then, the sensing topology of the group of agents is characterized by a fixed (time-invariant) directed graph $\mathcal{G} = (\mathcal{N}, \mathcal{A})$, where $\mathcal{A} \subset \mathcal{N} \times \mathcal{N}$, and the positive direction of arc $(i, j) \in \mathcal{A}$ is from i to j. We say that i can *affect* j (or, j can *sense* i) only if $(i, j) \in \mathcal{A}$. So, the arc direction of $\mathcal{A}$ is the direction of information flow. Let Π^i for all $i \in \mathcal{N}$ be the set of "neighbors" of agent i such that all $j \in \Pi^i$ can have their information about positions and velocities be sensed by agent i (but possibly with some delays or errors). Clearly, by definition, $(i, j) \in \mathcal{A}$ means $i \in \Pi^j$ (instead of $j \in \Pi^i$). Also, it can be that $i \in \Pi^j$ but $j \notin \Pi^i$ so that agent j can sense agent i but agent i cannot sense agent j. We assume $i \in \Pi^i$ since it is reasonable to assume that an agent knows its own position and velocity. We also assume each Π^i includes at least one $j \neq i$, that is, each agent has at least one such neighbor. Define $N^i = |\Pi^i|$ as the size of Π^i, so $2 \leq N^i \leq N$.

Agent to agent interactions considered here are of the "attract-repel" type where each agent seeks to be in a position that is "comfortable" relative to its neighbors [12]. Attraction here will be represented in u^i in a form like $-k'\left(x^i - x^j\right)$ where $k' > 0$ is a scalar that represents the strength of attraction. For repulsion, we use a repulsion term in u^i of the form

$$k_r \exp\left(\frac{-\frac{1}{2}\|x^i - x^j\|^2}{{r_s}^2}\right)\left(x^i - x^j\right) \tag{2}$$

where $k_r \geq 0$ and $r_s > 0$. Other types of attraction and repulsion terms are also possible.

We use a "resource profile" $J(x)$, $x \in \Re^n$, to represent the environment the agents move in. All agents move in the direction of $-\nabla J(x) = -\partial J/\partial x$, the negative gradient of $J(x)$, in order to move away from "bad" areas and into "good" areas of the environment [28]. There are many possible shapes for $J(x)$. We will study a family of profiles that are differentiable everywhere and $\|\nabla J(x)\| \leq \Lambda_0$ for all $x \in \Re^n$, with Λ_0 a known constant.

B. Sensing Delays, Noise, and Asynchronism

We assume that the ith agent, $i \in \mathcal{N}$, can sense the positions and velocities of all its neighbors, but with some time delay. In particular, let $\tau^i_j(k) \in [0, B_s] \subset \mathcal{Z}^+$ indicate the amount by which the position and velocity of agent j sensed by agent i at the kth step is outdated, with $B_s \in \mathcal{Z}^+$ a known constant. Thus, the position and velocity of agent j sensed by i at the kth step are written as $x^j(k - \tau^i_j(k))$ and $v^j(k - \tau^i_j(k))$, respectively. For simplicity of notation, hereafter we will write $\tau^i_j(k)$ as τ^i_j. But, it should be remembered that it is a *time-varying* integer-valued term. We will also assume that each agent can sense its own position and velocity without any delay, i.e., $\tau^i_i = 0$ for all i.

We assume that there exist sensing errors when each agent senses its own and other agents' positions and velocities. In particular, let $d^{ij}_p \in \Re^n$ and $d^{ij}_v \in \Re^n$ be these sensing errors (e.g., noise) for agent i with respect to agent j, respectively. Thus, if $j \in \Pi^i$, agent i actually senses agent j's position and velocity as $\hat{x}^j(k - \tau^i_j) = x^j(k - \tau^i_j) - d^{ij}_p(k)$ and $\hat{v}^j(k - \tau^i_j) = v^j(k - \tau^i_j) - d^{ij}_v(k)$. (In fact, precisely speaking, each agent i need not sense the *absolute* position and velocity but the *relative* ones of agent j. We will discuss more on this in Section II-C.) Notice that, different from time delays, we allow errors for an agent in sensing its own position and velocity, i.e., it could happen that $d^{ii}_p(k) \neq 0$ and $d^{ii}_v(k) \neq 0$. We assume the sensing error magnitudes are bounded by some constants, that is, $\|d^{ij}_p\| \leq \delta_p \in \Re^+$ and $\|d^{ij}_v\| \leq \delta_v \in \Re^+$ for all i and $j \in \Pi^i$, where δ_p and δ_v are known.

Besides the position and velocity sensing errors, we also assume there are some errors for agent i in sensing $\nabla J\left(x^i(k)\right)$, the gradient of the profile at its position at the kth step. Note that sensing errors related to profile could originate either from position sensing errors (i.e., $\hat{x}^i$ instead of x^i) or from gradient sensing errors (i.e., $\nabla\hat{J}$ instead of ∇J). For simplicity, we do not distinguish between these two cases and write the profile-related sensing error at the kth step as $d^i_f(k)$, i.e., we assume that $\nabla \hat{J}\left(\hat{x}^i(k)\right) = \nabla J(x^i(k)) - d_f^i(k)$. It is assumed that $\|d_f^i(k)\| \leq \delta_f \in \Re^+$ for all i, with δ_f a known constant. To simplify notation, we will write $\nabla \hat{J}\left(\hat{x}^i(k)\right)$ as $\nabla \hat{J}^i(k)$ from now on. We also assume $\nabla \hat{J}^i(k) = \nabla \hat{J}^i(0)$ for $k < 0, i \in \mathcal{N}$.

Next, we explain how our random delays lead to a type of asynchronous operation for our multiagent system. Of course, the motions of the agents are synchronous in the sense that the time step T is fixed. However, the time delays τ_j^i are random, but only fall on the boundaries of time intervals quantified by T. So, we are considering a restricted form of asynchronism, with B_sT the maximum sensing delay in real time. This is further illustrated by the following example. Suppose for agent i, $j_1 \in \Pi^i$, and $j_2 \in \Pi^i$. Also, suppose $B_s \geq 2$, and $\tau_{j_1}^i \in \{0,1\}$ and $\tau_{j_2}^i \in \{0,1,2\}$, with specific values from these sets chosen randomly at each k. It is possible that agent i senses agent j_1 according to $\tau_{j_1}^i(k) = 0,1,1,0,1,0,0,\ldots$, and agent j_2 according to $\tau_{j_2}^i(k) = 1,0,1,2,2,0,1,\ldots$, where both sequences are random. This means that agent i can sense the positions and velocities of agents j_1 and j_2 at different time steps. Thus, a restricted form of asynchronism is observed in this example.

C. Controls and Dynamics

Suppose that the general form of the control input for each agent i at the kth step is

$$\begin{aligned} u^i(k) = &- M_i k_p \sum_{j\in\Pi^i} \left(\hat{x}^i(k) - \hat{x}^j(k-\tau_j^i)\right) \\ &- M_i k_v \sum_{j\in\Pi^i} \left(\hat{v}^i(k) - \hat{v}^j(k-\tau_j^i)\right) - M_i k_d \hat{v}^i(k) \\ &+ M_i k_r \sum_{j\in\Pi^i} \exp\left(\frac{-\frac{1}{2}\left\|\hat{x}^i(k) - \hat{x}^j(k-\tau_j^i)\right\|^2}{{r_s}^2}\right) \\ &\times \left(\hat{x}^i(k) - \hat{x}^j(k-\tau_j^i)\right) - M_i k_f \nabla \hat{J}^i(k). \end{aligned} \tag{3}$$

Here, we think of the scalars $k_p > 0$ and $k_v > 0$ as the "attraction gains" which indicate how aggressive each agent is in aggregating. The gain $k_d > 0$ works as a "velocity damping gain." The gain $k_r \geq 0$ is a "repulsion gain" which sets how much that agent wants to be away from others and $r_s > 0$ represents its repulsion range. The gain $k_f \geq 0$ represents the strength of the agents' desire to move along the negative gradient of the resource profile.

By writing the control input as in (3), we are assuming that each agent i can sense its own position and velocity, but with some errors. Also, we assume agent i can sense the profile gradient at its position, but with some error. Recall in Section II-B, to initiate the description of our problem, we also assumed that agent i can sense the (absolute) positions and velocities (possibly with some delays and errors) of all its neighbor $j \in \Pi^i$. However, it should be noted that this is *not* required. In particular, only the *relative* position $\hat{x}^i(k) - \hat{x}^j(k-\tau_j^i)$ and the *relative* velocity $\hat{v}^i(k) - \hat{v}^j(k-\tau_j^i)$ need to be known by agent i, as one can see from (3).

Plugging (3) into (1) and eliminating all v-related terms by using $v(k) = (x(k+1) - x(k))/T$, we have

$$\begin{aligned} x^i(k+1) = &(2 - k_v N^i T - k_d T)x^i(k) \\ &+ (-1 - k_p N^i T^2 + k_v N^i T + k_d T)x^i(k-1) \\ &+ k_v T \sum_{j\in\Pi^i} x^j(k-\tau_j^i) + (k_p T^2 - k_v T) \\ &\times \sum_{j\in\Pi^i} x^j(k-1-\tau_j^i) + \psi^i(k-1)T^2 \\ &- k_f \nabla J^i(k-1)T^2 + \delta^i(k-1)T^2. \end{aligned} \tag{4}$$

where

$$\begin{aligned} \psi^i(k-1) = &k_r \sum_{j\in\Pi^i} \exp\left(-\frac{\left\|\hat{x}^i(k-1) - \hat{x}^j(k-1-\tau_j^i)\right\|^2}{2{r_s}^2}\right) \\ &\times \left(\hat{x}^i(k-1) - \hat{x}^j(k-1-\tau_j^i)\right) \end{aligned}$$

and

$$\begin{aligned} \delta^i(k-1) = &k_p \sum_{j\in\Pi^i} \left(d_p^{ii}(k-1) - d_p^{ij}(k-1)\right) \\ &+ k_v \sum_{j\in\Pi^i} \left(d_v^{ii}(k-1) - d_v^{ij}(k-1)\right) \\ &+ k_d d_v^{ii}(k-1) + k_f d_f^i(k-1). \end{aligned}$$

It is easy to show that the function $F(\psi) = \exp\left(-1/2\|\psi\|^2/{r_s}^2\right)\|\psi\|$, with ψ any real vector, has a unique maximum value of $\exp(-1/2)r_s$ which is achieved when $\|\psi\| = r_s$ [11]. Then, for any i and k, we have

$$\|\psi^i(k)\| \leq (N^i - 1)\exp\left(-\frac{1}{2}\right)k_r r_s \leq \psi_0 \tag{5}$$

where we used the fact that $x^i(k) - x^i(k-\tau_i^i) = 0$, and $\psi_0 = (N-1)\exp(-1/2)k_r r_s$. Since for any given vector z we have $\max_j |z_j| \leq \|z\|$, $\max_j |\psi_j^i(k)| \leq \psi_0$. Thus, for all i and k we have

$$|\psi^i(k)| \leq \bar{\psi} = \psi_0 \mathbf{1}_n \in \Re^n. \tag{6}$$

Sometimes we know a constant integer $\bar{N}$ such that $N^i \leq \bar{N} \leq N$ for all i. Then, let $\psi_0' = (\bar{N} - 1)\exp(-1/2)k_r r_s$, and we have

$$|\psi^i(k)| \leq \bar{\psi}' = \psi_0' \mathbf{1}_n \in \Re^n. \tag{7}$$

Generally, using (7) instead of (6) helps alleviate conservativeness in the results, especially when $\bar{N} \ll N$. Nevertheless, for brevity, we use (6) throughout the paper.

Similarly, recall $\|\nabla J(x)\| \leq \Lambda_0$ by assumption, for all i and k we have

$$|k_f \nabla J^i(k)| \leq \bar{\Lambda} = k_f \Lambda_0 \mathbf{1}_n \in \Re^n. \tag{8}$$

Denote $\Delta_0 = 2k_p N\delta_p + (2k_v N + k_d)\delta_v + k_f \delta_f$. Then, for the last term in (4) we have

$$|\delta^i(k-1)| \leq \bar{\Delta} = \Delta_0 \mathbf{1}_n \in \Re^n. \tag{9}$$

III. Stability Analysis of Cohesion Properties

In this section, we give the main results on stability analysis of cohesion. In Section III-A we specify some mathematical properties (Lemmas 1 and 2) and an assumption that we use in the remainder of the paper. In Section III-B we quantify several properties of the dynamics of agent position trajectories relative to the sensing topology. Specifically, we first define some measures on the state of the system and show that the increment rates of these measures are bounded (Lemma 3), and then we show that (Lemmas 4 to 6) the difference between the upper and lower boundaries of the measures decreases (componentwise) at the rate of a geometric progression, but with some "perturbation" term. In Section III-C, we start by defining an error coordinate system. Then based on the results from Section III-B, we provide our main results (Theorem 1 and 2), which indicate that the system trajectories are uniformly ultimately bounded in the error coordinate system, and that under certain conditions the set of zero interagent distances is exponentially stable. Finally, in Section III-D, we interpret the results and give insights into how system parameters and information flow constraints affect cohesiveness.

A. Mathematical Preliminaries

Let $c_1 = 2 - k_v NT - k_d T$, $c_2 = -1 - k_p NT^2 + k_v NT + k_d T$, $c_3 = k_v T$, and $c_4 = k_p T^2 - k_v T$. Let $c_1^i = 2 - k_v N^i T - k_d T$ and $c_2^i = -1 - k_p N^i T^2 + k_v N^i T + k_d T$, $i \in \mathcal{N}$. Also, denote $\Theta^i(k) = \psi^i(k) - k_f \nabla J^i(k) + \delta^i(k)$. Then, we can write (4) as

$$x^i(k+1) = c_1^i x^i(k) + c_2^i x^i(k-1) + c_3 \sum_{j \in \Pi^i} x^j(k - \tau_j^i) + c_4 \sum_{j \in \Pi^i} x^j(k - 1 - \tau_j^i) + \Theta^i(k-1)T^2. \quad (10)$$

Let $\Theta_0 = \psi_0 + k_f \Lambda_0 + \Delta_0$, with ψ_0, Λ_0, and Δ_0 defined in (5), (8), and (9), respectively. Also, let $\bar{\Theta} = \bar{\psi} + \bar{\Lambda} + \bar{\Delta}$. Obviously, $\bar{\Theta} = \Theta_0 \mathbf{1}_n$, and $|\Theta^i(k)| \leq \bar{\Theta}$ for all i and k.

Next, we present an assumption which will be used throughout the remainder of this paper.

Assumption 1: The system parameters k_p, k_v, k_d, N, and T are such that $c_1 > 0$, $c_2 > 0$, $c_3 > 0$, and $c_4 \geq 0$.

The following two Lemmas which will be useful in our proofs.

Lemma 1: If Assumption 1 holds, then *i)* $c_1 \in (0,1)$, $c_2 \in (0,1)$, $c_3 \in (0,1)$, and $c_4 \in [0,1)$; and *ii)* $c_1^i \in [c_1, 1)$ and $c_2^i \in [c_2, 1)$ for all $i \in \mathcal{N}$.

Proof: Note that $c_1 + c_2 + c_3 N + c_4 N = 1$, we have *i)* immediately. Also, note that $c_1^i + c_2^i + c_3 N^i + c_4 N^i = 1$ and $N^i \leq N$ for all i, we have *ii)* holds. ■

Lemma 2: Suppose u, v, w and z are real vectors of the same dimension, and z is nonnegative componentwise. If $u \geq \min\{v, w\} + z$, then $\min\{u, v\} \geq \min\{v, w\} - z$. Also, if $u \leq \max\{v, w\} + z$, then $\max\{u, v\} \leq \max\{v, w\} + z$.

Proof: For the first part, consider the ith entry of each vector, with i picked arbitrarily. Then, we have $u_i \geq \min\{v_i, w_i\} + z_i$. It can be shown that $\min\{u_i, v_i\} \geq \min\{v_i, w_i\} - z_i$, either $v_i < w_i$ or $v_i \geq w_i$. Since i is picked arbitrarily, we have $\min\{u, v\} \geq \min\{v, w\} - z$. The second part is proven similarly. ■

B. Properties of Agent Position Dynamics Relative to Sensing Topology

Define the state $X(k) \in \Re^{Nn \times (B_s+2)}$ for the system as

$$X(k) = \begin{bmatrix} x^{11}(k) & x^{12}(k) & \dots & x^{1(B_s+2)}(k) \\ x^{21}(k) & x^{22}(k) & \dots & x^{2(B_s+2)}(k) \\ \vdots & \vdots & & \vdots \\ x^{N1}(k) & x^{N2}(k) & \dots & x^{N(B_s+2)}(k) \end{bmatrix} \quad (11)$$

where $x^{ij}(k) = x^i(k - j + 1) \in \Re^n$, with $i \in \mathcal{N}$ and $j \in \mathcal{B}$, $\mathcal{B} = \{1, \ldots, B_s + 2\}$. Define $M(k) \in \Re^n$, the maximum displacement from the origin of any agent over the last $B_s + 2$ time steps in each dimension, as

$$M(k) = \max_{i \in \mathcal{N}} \left\{ \max_{j \in \mathcal{B}} \left\{ x^{ij}(k) \right\} \right\} = \max_{i \in \mathcal{N}} \left\{ \max_{0 \leq \tau \leq B_s+1} \left\{ x^i(k - \tau) \right\} \right\}. \quad (12)$$

Similarly, let $m(k) \in \Re^n$ be the minimum such displacement, defined as

$$m(k) = \min_{i \in \mathcal{N}} \left\{ \min_{j \in \mathcal{B}} \left\{ x^{ij}(k) \right\} \right\} = \min_{i \in \mathcal{N}} \left\{ \min_{0 \leq \tau \leq B_s+1} \left\{ x^i(k - \tau) \right\} \right\}. \quad (13)$$

First, we show that the rates of increase (decrease) of elements of $M(k)$ (respectively, $m(k)$) are bounded.

Lemma 3: If Assumption 1 holds, then for any integer n_2 and n_1, with $n_2 \geq n_1 > 0$, we have

$$\begin{aligned} M(n_2) &\leq M(n_1) + (n_2 - n_1)\bar{\Theta}T^2 \\ m(n_2) &\geq m(n_1) - (n_2 - n_1)\bar{\Theta}T^2 \end{aligned} \quad (14)$$

with $\bar{\Theta}$ defined in Section III-A.

Proof: If Assumption 1 holds, then by Lemma 1, $c_1^i > 0$ and $c_2^i > 0$. Since $\tau_j^i \in [0, B_s]$, by the definition of $M(k)$, we have from (10) that

$$\begin{aligned} x^i(k+1) &\leq c_1^i M(k) + c_2^i M(k) + c_3 \sum_{j \in \Pi^i} M(k) \\ &\quad + c_4 \sum_{j \in \Pi^i} M(k) + \Theta^i(k-1)T^2 \\ &\leq M(k) + \bar{\Theta}T^2 \end{aligned}$$

for all $k \geq 0$, where we used the fact that $c_1^i + c_2^i + c_3 N^i + c_4 N^i = 1$ and $|\Theta^i(k)| < \bar{\Theta}$. Since the previous equation holds for all i, we have

$$\begin{aligned} &\max_{i \in \mathcal{N}} \left\{ x^i(k+1) \right\} \\ &\leq \max \left\{ \max_{i \in \mathcal{N}} \left\{ \max_{0 \leq \tau \leq B_s} \left\{ x^i(k - \tau) \right\} \right\}, \max_{i \in \mathcal{N}} \left\{ x^i(k - 1 - B_s) \right\} \right\} \\ &\quad + \bar{\Theta}T^2. \end{aligned}$$

With Lemma 2, we have from the aforementioned equation that

$$\max\left\{\max_{i\in\mathcal{N}}\left\{x^i(k+1)\right\},\max_{i\in\mathcal{N}}\left\{\max_{0\le\tau\le B_s}\left\{x^i(k-\tau)\right\}\right\}\right\}$$
$$\le\max\left\{\max_{i\in\mathcal{N}}\left\{\max_{0\le\tau\le B_s}\left\{x^i(k-\tau)\right\}\right\},\max_{i\in\mathcal{N}}\left\{x^i(k-1-B_s)\right\}\right\}+\bar{\Theta}T^2.$$

By definition of $M(k)$, this is

$$M(k+1)\le M(k)+\bar{\Theta}T^2. \tag{15}$$

Now, for the case of $n_2>n_1>0$, we have by using (15) repeatedly

$$M(n_2)\le M(n_1)+(n_2-n_1)\bar{\Theta}T^2.$$

Since the previous equation also holds when $n_2=n_1$, we have $M(n_2)\le M(n_1)+(n_2-n_1)\bar{\Theta}T^2$ for all $n_2\ge n_1>0$.

Similarly, we can show that $m(n_2)\ge m(n_1)-(n_2-n_1)\bar{\Theta}T^2$ for $n_2\ge n_1>0$. ∎

Next, we will prove three Lemmas, with our ultimate objective to show that there exist constants $\eta_1>0$, $\eta_1\in\mathcal{Z}^+$, $\eta_2\in(0,1)\subset\Re$ and $\eta_3\in\Re^+$ such that $M(k+\eta_1)-m(k+\eta_1)\le(1-\eta_2)(M(k)-m(k))+\eta_3\mathbf{1}_n$ for all $k\in\mathcal{Z}^+$. This shows that $M(k)-m(k)$ decreases (componentwise) at the rate of a geometric progression, but with some "perturbation" term characterized by η_3. In proving these Lemmas, we use Lemma 3 and ideas similar to those in [23]. Next, we present another assumption, a standard one for consensus problems [23], which is needed by the following Lemmas.

Assumption 2: There exists a fixed nonempty set $\mathcal{D}\subseteq\mathcal{N}$ of "distinguished" agents such that for every $i\in\mathcal{D}$ and every $j\in\mathcal{N}$, there exists a positive path from i to j in the directed graph $\mathcal{G}$, defined in Section II-A.

Recall that in Section II-A we define the concept of sensing topology and the corresponding directed graph $\mathcal{G}$ of our system, but the connectivity of the topology is not clearly defined. As expected, such connectivity cannot be arbitrary. Assumption 2 gives a constraint on it. This assumption means that the position and velocity information of every distinguished agent i can, through certain positive paths, affect every agent j, even if i and j are not directly connected to each other. Note that for a graph, having such a distinguished set is a much milder condition than being completely connected.

Throughout the next three Lemmas, we fix some $l\in\mathcal{D}$. Define D_p, $1\le p\le N-1$, as the set of $i\in\mathcal{N}$ and $i\ne l$ such that p is the *minimum* number of arcs in a positive path from l to i in $\mathcal{G}$. Also, let $D_0=\{l\}$. Then, there exists an integer P, $1\le P\le N-1$, such that $D_0\bigcup D_1\bigcup\ldots\bigcup D_P=\mathcal{N}$ and every $i\in\mathcal{N}$ belongs (and only belongs) to one of the sets $D_0,D_1,\ldots,D_P$. Also, for every $i\in D_p, p\in\{1,\ldots,P\}$, there must exist some $j\in D_{p-1}$ such that, by the definition of $\mathcal{A}$ in Section II-A, $(j,i)\in\mathcal{A}$, i.e., i can sense j. One illustrative example is given as follows. Fig. 1 shows the sensing topology of five agents, where the direction of each arrow indicates the positive direction of an arc in $\mathcal{A}$. For example, arc $(1,2)\in\mathcal{A}$, meaning the position and velocity information of agent 1 affects agent 2 (or, agent 2 can sense agent 1). Arc $(i,i)\in\mathcal{A}$, $i\in\{1,\ldots,5\}$, since all agents can sense their own positions and velocities. Moreover, agents 1 and 5 can sense each other, and so can agents 2 and 4. Also, by Assumption 2, $\mathcal{D}=\{1,5\}$. If we choose $l=1$, then $P=2$, $\mathcal{D}_0=\{1\}$, $\mathcal{D}_1=\{2,5\}$, and $\mathcal{D}_2=\{3,4\}$. If we choose $l=5$, then $P=3$, $\mathcal{D}_0=\{5\}$, $\mathcal{D}_1=\{1\}$, $\mathcal{D}_2=\{2\}$, and $\mathcal{D}_3=\{3,4\}$.

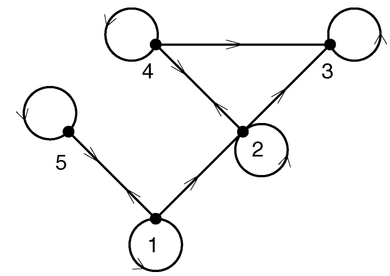


Fig. 1. Simple sensing topology with five agents.

In the next lemma, we will show that in a n_0-indexed fixed-length time slot $\mathcal{K}_0\subset\mathcal{Z}^+$, with n_0 any nonnegative integer, the position trajectory of agent l, $l\in\mathcal{D}$, is bounded in terms of $M(n_0)$, $m(n_0)$, $x^l(n_0)$, and system parameters.

Lemma 4: If Assumptions 1 and 2 hold, then for any agent $l\in\mathcal{D}$, and for every $k\in\mathcal{K}_0=[n_0,\ n_0+(P+1)(B_s+1)]$, with any $n_0\in\mathcal{Z}^+$, we have

$$\begin{aligned}x^l(k+1)&\le M(n_0)-\alpha\left(M(n_0)-x^l(n_0)\right)-\beta_1\gamma\bar{\Theta}T^2\\ x^l(k+1)&\ge m(n_0)+\alpha\left(x^l(n_0)-m(n_0)\right)+\beta_1\gamma\bar{\Theta}T^2\end{aligned} \tag{16}$$

where $\alpha=c_1^{(P+1)(B_s+1)+1}$, $\beta_1=(1-k_vNT-k_dT)(P+1)(B_s+1)-2+k_dT$, and

$$\gamma=\frac{1-(2-k_dT)^{(P+1)(B_s+1)+1}}{k_dT-1}.$$

Proof: Knowing $c_1^l+c_2^l+c_3N^l+c_4N^l=1$, we have from (10)

$$\begin{aligned}M(n_0)x^l(k+1)&\\=c_1^l(M(n_0)-x^l(k))&+c_2^l(M(n_0)-x^l(k-1))\\&+c_3\sum_{j\in\Pi^l}\left(M(n_0)-x^j(k-\tau_j^l)\right)\\&+c_4\sum_{j\in\Pi^l}\left(M(n_0)-x^j(k-1-\tau_j^l)\right)-\Theta^l(k-1)T^2.\end{aligned} \tag{17}$$

Moreover, note that $1-k_dT<0$ since $c_2^l+c_4N^l>0$. Thus, $(1-k_dT)((P+1)(B_s+1)-1)<0<k_vNT(P+1)(B_s+1)$, that is, $\beta_1=(1-k_vNT-k_dT)(P+1)(B_s+1)-2+k_dT<-1$. Since $c_1+c_3N>0$, we have $2-k_dT\in(0,1)$ and, thus, $\gamma=\left(1-(2-k_dT)^{(P+1)(B_s+1)+1}\right)/(1-(2-k_dT))>1$. Therefore, $\beta_1\gamma<\beta_1<-1$. In the proof, we consider two cases separately: $k=n_0$ and $k\in[n_0+1,n_0+(P+1)(B_s+1)]$.

Case (i): $\mathbf{k=n_0}$*:* When $k=n_0$, all x-related terms on the right-hand side of (17) have time indices in the range of $[n_0-B_s-1,n_0]$ (where n_0-B_s-1 could be negative) and thus, are less than or equal to $M(n_0)$ by definition. Therefore, (17) can be immediately written as

$$M(n_0) - eex^l(k+1) \geq c_1^l(M(n_0) - x^l(k)) - \bar{\Theta}T^2 \geq c_1^l(M(n_0)-x^l(k))+\beta_1\bar{\Theta}T^2. \quad (18)$$

Since $c_1^l \geq c_1 > \alpha$ and $\beta_1\gamma < \beta_1 < -1$, (16) holds for this case.

Case ii): $\mathbf{k} \in [\mathbf{n_0+1}, \mathbf{n_0+(P+1)(B_s+1)}]$*:* Before we proceed, for each fixed l and n_0 define two sets $\Pi_L^l(k)$ and $\Pi_G^l(k)$, which will be used in our deduction. Specifically, let

$$\Pi_L^l(k) = \{j : k - \tau_j^l \leq n_0,\ j \in \Pi^l\}$$

which is the set of agents that satisfy $x^j(k-\tau_j^l) \leq M(n_0)$ by the definition of $M(k)$, and let

$$\Pi_G^l(k) = \{j : k - \tau_j^l > n_0,\ j \in \Pi^l\}$$

which is the set of agents that, though they may or may not satisfy $x^j(k-\tau_j^l) \leq M(n_0)$, have Lemma 3 applicable to $M(k-\tau_j^l)$ and $M(n_0)$ with $n_2 = k - \tau_j^l$ and $n_1 = n_0$. Also, let $N_L^l(k)$ and $N_G^l(k)$ be the sizes of $\Pi_L^l(k)$ and $\Pi_G^l(k)$, respectively. Then, $N_L^l(k) + N_G^l(k) = N^l$. Obviously, for all $j \in \Pi_L^l(k)$ we have $k - 1 - \tau_j^l < n_0$ (and thus, $x^j(k - 1 - \tau_j^l) \leq M(n_0)$ by definition), and for all $j \in \Pi_G^l(k)$ we have $k - 1 - \tau_j^l \geq n_0$ (and, thus, we can apply Lemma 3 to $M(k-1-\tau_j^l)$ and $M(n_0)$) since all the numbers involved here are integers. These facts will be used in our deduction. Note that we do not know the explicit sets $\Pi_L^l(k)$ and $\Pi_G^l(k)$; all we know is that they exist for any $k \in \mathcal{Z}^+$. The explicit values in the sets clearly depend on k but we allow k to be arbitrary in the time slot $[n_0+1, n_0+(P+1)(B_s+1)]$, so the analysis that follows is valid for all k in this time slot.

Now, (17) can be rewritten as

$$\begin{aligned} M(n_0) - x^l(k+1) &= c_1^l(M(n_0) - x^l(k)) + c_2^l(M(n_0) - x^l(k-1)) \\ &+ c_3 \sum_{j\in\Pi_G^l(k)} \left(M(n_0) - x^j(k-\tau_j^l)\right) \\ &+ c_3 \sum_{j\in\Pi_L^l(k)} \left(M(n_0) - x^j(k-\tau_j^l)\right) \\ &+ c_4 \sum_{j\in\Pi_G^l(k)} \left(M(n_0) - x^j(k-1-\tau_j^l)\right) \\ &+ c_4 \sum_{j\in\Pi_L^l(k)} \left(M(n_0) - x^j(k-1-\tau_j^l)\right) - \Theta^l(k-1)T^2. \end{aligned}$$

By definition of $\Pi_L^l(k)$ and $M(k)$, the $j \in \Pi_L^l(k)$ terms are nonnegative. Thus

$$\begin{aligned} M(n_0) - x^l(k+1) &\geq c_1^l(M(n_0) - x^l(k)) + c_2^l(M(n_0) - M(k-1)) \\ &+ c_3 \sum_{j\in\Pi_G^l(k)} \left(M(n_0) - M(k-\tau_j^l)\right) \\ &+ c_4 \sum_{j\in\Pi_G^l(k)} \left(M(n_0) - M(k-1-\tau_j^l)\right) - \bar{\Theta}T^2. \end{aligned}$$

Next, using Lemma 3 and the fact that $k \geq n_0 + 1$, we have

$$\begin{aligned} M(n_0) - x^l(k+1) &\geq c_1^l(M(n_0) - x^l(k)) + c_2^l(n_0 - k + 1)\bar{\Theta}T^2 \\ &+ c_3 \sum_{j\in\Pi_G^l(k)} (n_0 - k + \tau_j^l)\bar{\Theta}T^2 \\ &+ c_4 \sum_{j\in\Pi_G^l(k)} (n_0 - k + 1 + \tau_j^l)\bar{\Theta}T^2 - \bar{\Theta}T^2 \\ &\geq c_1^l(M(n_0) - x^l(k)) + \left[c_2^l(n_0 - k + 1) \right. \\ &\left. + c_3N_G^l(k)(n_0-k) + c_4N_G^l(k)(n_0 - k + 1) - 1\right] \bar{\Theta}T^2 \\ &\geq c_1^l(M(n_0) - x^l(k)) + \left[c_2^l(n_0 - k + 1) \right. \\ &\left. + c_3N^l(n_0 - k) + c_4N^l(n_0 - k + 1) - 1\right] \bar{\Theta}T^2. \end{aligned}$$

Using the assumption of $k \leq n_0 + (P+1)(B_s+1)$ and the fact that $N^l \leq N$, we have after some manipulations

$$M(n_0) - x^l(k+1) \geq c_1^l(M(n_0) - x^l(k)) + \beta_1\bar{\Theta}T^2. \quad (19)$$

Recall $k \in [n_0+1,\ n_0+(P+1)(B_s+1)]$ in (19). For now, consider $k = n_0 + 1$, then from (19) we have

$$M(n_0)-x^l(n_0+2) \geq c_1^l(M(n_0)-x^l(n_0+1))+\beta_1\bar{\Theta}T^2. \quad (20)$$

Since (18) gives

$$M(n_0) - x^l(n_0+1) \geq c_1^l(M(n_0) - x^l(n_0)) + \beta_1\bar{\Theta}T^2$$

we have from (20) that

$$M(n_0) - x^l(n_0+2) \geq (c_1^l)^2(M(n_0) - x^l(n_0)) + c_1^l\beta_1\bar{\Theta}T^2 + \beta_1\bar{\Theta}T^2.$$

Repeating this procedure for all $k \in [n_0+1,\ n_0+(P+1)(B_s+1)]$ and using the facts that $c_1 \leq c_1^l \leq 2 - k_dT < 1$ and $\beta_1 < 0$, we have

$$\begin{aligned} M(n_0) - x^l(k+1) &\geq (c_1^l)^{(P+1)(B_s+1)+1}(M(n_0)-x^l(n_0)) \\ &+ \sum_{z=0}^{(P+1)(B_s+1)} (c_1^l)^z\beta_1\bar{\Theta}T^2 \geq c_1^{(P+1)(B_s+1)+1}(M(n_0) - x^l(n_0)) \\ &+ \sum_{z=0}^{(P+1)(B_s+1)} (2-k_dT)^z\beta_1\bar{\Theta}T^2. \end{aligned}$$

Finally, in the previous equation, using the fact that for a geometric series, with $c \in (0,1)$, the sum

$$\sum_{z=0}^{m} c^z = \frac{(1-c^{m+1})}{(1-c).} \quad (21)$$

we prove the first part of the lemma. The second part is proven similarly. ■

So far, we have shown that in an n_0-indexed time slot $\mathcal{K}_0$, the position trajectory of any agent $l \in \mathcal{D}$ is bounded in terms of $M(n_0)$, $m(n_0)$, $x^l(n_0)$ and system parameters. In the next lemma, we will show that for the same agent l as in Lemma 4, for any agent $i \in \mathcal{N}$ there exists a $p \in \{0, 1, \ldots, P\}$ and correspondingly, an n_0-indexed fixed-length time slot $\mathcal{K}_p \subseteq \mathcal{K}_0$, such that in $\mathcal{K}_p$ the position trajectory of agent i is bounded in terms of $M(n_0)$, $m(n_0)$, $x^l(n_0)$ and system parameters, n_0 as defined in Lemma 4.

Lemma 5: If Assumptions 1 and 2 hold, then for every $p \in \{0, 1, \dots, P\}$, there exists some α_p such that for every $k \in \mathcal{K}_p = [n_0 + p(B_s+1),\ n_0+(P+1)(B_s+1)]$, with $n_0 \in \mathcal{Z}^+$, and for every $i \in D_p$, we have

$$x^i(k+1) \le M(n_0) - \alpha_p\left(M(n_0) - x^l(n_0)\right) - \gamma_p\bar{\Theta}T^2$$
$$x^i(k+1) \ge m(n_0) + \alpha_p\left(x^l(n_0) - m(n_0)\right) + \gamma_p\bar{\Theta}T^2. \quad (22)$$

where $\alpha_p = c_3^p\alpha$ and $\gamma_p = c_3^p\beta_1\gamma + \sum_{z=0}^{p-1} c_3^z\beta_2$ (and particularly, $\gamma_0 = \beta_1\gamma$), with α, β_1 and γ as defined in Lemma 4, and $\beta_2 = -(1-k_vT)(P+1)(B_s+1) - 2 + k_dT < 0$.

Proof: We show this by induction. From Lemma 4, obviously this holds for $p = 0$ case, where $i = l$ and $D_0 = \{l\}$. Suppose it holds for the case of $p = p'$, $1 \le p' < P$, then we need to show that it also holds for any $i \in D_{p'+1}$. By Assumption 2, for any $i \in D_{p'+1}$ there must exist some $i' \in D_{p'}$ such that $i' \in \Pi^i$. Note that for i', by the induction hypothesis, it satisfies

$$x^{i'}(k'+1) \le M(n_0) - \alpha_{p'}\left(M(n_0) - x^l(n_0)\right) - \gamma_{p'}\bar{\Theta}T^2 \quad (23)$$

where $k' \in \mathcal{K}_{p'} = [n_0 + p'(B_s+1),\ n_0+(P+1)(B_s+1)]$.

Now, for $k \in \mathcal{K}_{p'+1} = [n_0 + (p'+1)(B_s+1),\ n_0 + (P+1)(B_s+1)]$, we have from (10)

$$\begin{aligned} &M(n_0) - x^i(k+1) \\ &\quad = c_1^i(M(n_0) - x^i(k)) + c_2^i(M(n_0) - x^i(k-1)) \\ &\qquad + c_3\sum_{j\in\Pi^i}\left(M(n_0) - x^j(k-\tau_j^i)\right) \\ &\qquad + c_4\sum_{j\in\Pi^i}\left(M(n_0) - x^j(k-1-\tau_j^i)\right) - \Theta^i(k-1)T^2 \\ &\quad \ge c_1^i(M(n_0) - M(k)) + c_2^i(M(n_0) - M(k-1)) \\ &\qquad + c_3\sum_{j\in\Pi^i, j\ne i'}\left(M(n_0) - M(k-\tau_j^i)\right) \\ &\qquad + c_3(M(n_0) - x^{i'}(k-\tau_{i'}^i)) \\ &\qquad + c_4\sum_{j\in\Pi^i}\left(M(n_0) - M(k-1-\tau_j^i)\right) - \bar{\Theta}T^2. \end{aligned}$$

Since $k \in \mathcal{K}_{p'+1}$, $n_0 < k - \tau_j^i$ for all possible τ_j^i. By Lemma 3

$$\begin{aligned} &M(n_0) - x^i(k+1) \\ &\quad \ge c_1^i(n_0-k)\bar{\Theta}T^2 + c_2^i(n_0-k+1)\bar{\Theta}T^2 \\ &\qquad + c_3\sum_{j\in\Pi^i, j\ne i'}(n_0-k+\tau_j^i)\bar{\Theta}T^2 \\ &\qquad + c_3(M(n_0) - x^{i'}(k-\tau_{i'}^i)) \\ &\qquad + c_4\sum_{j\in\Pi^i}(n_0-k+1+\tau_j^i)\bar{\Theta}T^2 - \bar{\Theta}T^2 \\ &\quad \ge c_3(M(n_0) - x^{i'}(k-\tau_{i'}^i)) + [c_1^i(n_0-k) \\ &\qquad + c_2^i(n_0-k+1) + c_3(N^i-1)(n_0-k) \\ &\qquad + c_4N^i(n_0-k+1) - 1]\bar{\Theta}T^2. \end{aligned}$$

Note that $k \in \mathcal{K}_{p'+1}$, so $n_0 - k \ge -(P+1)(B_s+1)$ and, thus, the previous equation can be written as

$$M(n_0) - x^i(k+1) \ge c_3(M(n_0) - x^{i'}(k-\tau_{i'}^i)) + \beta_2\bar{\Theta}T^2. \quad (24)$$

Let $\tilde{k} = k'+1$. Recall $k' \in \mathcal{K}_{p'}$, then $\tilde{k} \in \tilde{\mathcal{K}}_{p'} = [n_0 + p'(B_s+1)+1, n_0+(P+1)(B_s+1)+1]$. Rewrite (23) as

$$M(n_0) - x^{i'}(\tilde{k}) \ge c_3^{p'}\alpha\left(M(n_0) - x^l(n_0)\right) + c_3^{p'}\beta_1\gamma\bar{\Theta}T^2 + \sum_{z=0}^{p'-1} c_3^z\beta_2\bar{\Theta}T^2. \quad (25)$$

By assumption, $k \in \mathcal{K}_{p'+1} = [n_0 + (p'+1)(B_s+1),\ n_0 + (P+1)(B_s+1)]$, therefore, in (24) we have

$$k - \tau_{i'}^i \in \tilde{\mathcal{K}}_{p'+1} = [n_0 + p'(B_s+1)+1, n_0+(P+1)(B_s+1)] \subset \tilde{\mathcal{K}}_{p'}.$$

Since (25) is valid on $\tilde{\mathcal{K}}_{p'}$, it must also be valid on $\tilde{\mathcal{K}}_{p'+1}$. Thus, we can plug (25) into (24) directly by replacing $x^{i'}(\tilde{k})$ with $x^{i'}(k-\tau_{i'}^i)$, and this gives

$$M(n_0) - x^i(k+1) \ge c_3^{p'+1}\alpha(M(n_0) - x^l(n_0)) + c_3^{p'+1}\beta_1\gamma\bar{\Theta}T^2 + \sum_{z=0}^{p'} c_3^z\beta_2\bar{\Theta}T^2.$$

This proves the first part of the Lemma. The second part is proven similarly. ■

Intuitively, Lemma 4 means that if agent $l \in \mathcal{D}$ has its position and velocity changed at any time step $n_0 \in \mathcal{Z}^+$, then its position trajectory will be affected by that change thereafter since it can sense its own information without delay, while in comparison, Lemma 5 means that, when agent l changes its position and velocity at n_0, for an agent $i \in \mathcal{D}_p$ (meaning it is p arcs away from agent l on the topology), possibly it cannot be affected by this change until time step $n_0 + p(B_s+1)$ since the sensing delay on each arc could be up to $B_s + 1$ steps.

Notice that $\mathcal{K}_P \subset \mathcal{K}_{P-1} \subset \cdots \subset \mathcal{K}_0$. Thus, given an agent $l \in \mathcal{D}$, Lemma 5 holds in time slot $\mathcal{K}_P$ for any agent $i \in \mathcal{N}$. This means that with respect to agent l, the position trajectory of any agent $i \in \mathcal{N}$ is bounded in $\mathcal{K}_P$ in terms of $M(n_0)$, $m(n_0)$, $x^l(n_0)$, and system parameters. This gives us the following Lemma, which, as we stated earlier, shows that $M(k) - m(k)$ decreases at the rate of a geometric progression, but with some "perturbation" term.

Lemma 6: If Assumptions 1 and 2 hold, then for every $n_0 \in \mathcal{Z}^+$, we have

$$\begin{aligned} &M(n_0 + (P+1)(B_s+1)+1) \\ &\quad - m(n_0 + (P+1)(B_s+1)+1) \\ &\quad \le (1 - c_3^P\alpha)(M(n_0) - m(n_0)) + 2\beta\bar{\Theta}T^2 \end{aligned} \quad (26)$$

where $\beta = \max\{-\gamma_0, -\gamma_P\} > 0$, with γ_p defined in Lemma 5.

Proof: We first show that $\beta = \max_{0\le p\le P}\{-\gamma_p\}$. To see this, pick any $p \in \{1, \dots, P-1\}$, then from the parameters defined in Lemma 5 and (21), after some manipulations we have

$$\gamma_{p+1} - \gamma_p = c_3^p\left((c_3-1)\beta_1\gamma + \beta_2\right) = c_3^p\left(\gamma_1 - \gamma_0\right).$$

Thus, if $\gamma_1 \ge \gamma_0$, then $\gamma_0 = \min_{0\le p\le P}\{\gamma_p\}$, otherwise $\gamma_P = \min_{0\le p\le P}\{\gamma_p\}$. Therefore, $\beta = \max\{-\gamma_0, -\gamma_P\} = \max_{0\le p\le P}\{-\gamma_p\}$. Notice that $\beta > 0$ since $\gamma_p < 0$ for all p.

Now, given $\mathcal{K}_P = [n_0 + P(B_s+1),\ n_0 + (P+1)(B_s+1)]$, for every $n_0 \in \mathcal{Z}^+$ we have by the definition of $M(k)$

$$M(n_0 + (P+1)(B_s+1)+1) = \max_{i\in\mathcal{N}}\left\{\max_{k\in\mathcal{K}_P}\{x^i(k+1)\}\right\}.$$

Note that for each $x^i(k+1)$, $i\in\mathcal{N}$ and $k\in\mathcal{K}_P$, from Lemma 5 we have for the fixed $l\in\mathcal{D}$ that

$$x^i(k+1) \le M(n_0) - c_3^P\alpha(M(n_0) - x^l(n_0)) + \beta\bar{\Theta}T^2$$

where we used the facts that $c_3^P\alpha = \min_{0\le p\le P}\{\alpha_p\}$ and $\beta = \max_{0\le p\le P}\{-\gamma_p\}$. Therefore

$$\begin{aligned}M(n_0 + (P+1)(B_s+1)+1)\\ \le M(n_0) - c_3^P\alpha(M(n_0) - x^l(n_0)) + \beta\bar{\Theta}T^2.\end{aligned}$$

Similarly

$$\begin{aligned}m(n_0 + (P+1)(B_s+1)+1)\\ \ge m(n_0) + c_3^P\alpha(x^l(n_0) - m(n_0)) - \beta\bar{\Theta}T^2.\end{aligned}$$

Subtracting these two equations, we obtain (26). ■

Note that $2\beta\bar{\Theta}T^2$ is only affected by system parameters, sensing errors and resource profiles and thus, is bounded. So from (26), intuitively we can see that when $M(n_0) - m(n_0)$ is sufficiently large, the first term on the right-hand side of (26) will dominate the second term and thus, $M(n_0) - m(n_0)$ will decrease in time. Next, we will quantify this idea and provide a uniform ultimate bound on the position trajectories.

C. Cohesion: Uniform Ultimate Boundedness and Exponential Stability

Before we proceed, we define an error coordinate system. Let

$$\begin{aligned}\bar{x}(k) &= \frac{1}{N(B_s+2)}\sum_{i=1}^{N}\sum_{j=1}^{B_s+2} x^{ij}(k)\\ &= \frac{1}{N(B_s+2)}\sum_{i=1}^{N}\sum_{\tau=0}^{B_s+1} x^{i}(k-\tau) \qquad (27)\end{aligned}$$

be the averaged centroid position of all the agents during the last B_s+2 time steps, with $x^{ij}(k)$ as defined in (11). By definition, $m(k)\le\bar{x}(k)\le M(k)$. Define $e^{ij}(k) = x^{ij}(k) - \bar{x}(k) = x^i(k-j+1) - \bar{x}(k) \in \Re^n$ as the position error (i.e., relative displacement) of agent i with respect to $\bar{x}(k)$ at time step $k-j+1$, with $i\in\mathcal{N}$ and $j\in\mathcal{B}$. Thus, e_l^{ij}, $l=1,\dots,n$, is the lth component of the error vector. In this error coordinate system, we define $E(k)\in\mathcal{E} = \Re^{Nn\times(B_s+2)}$ for the system as

$$E(k) = \begin{bmatrix} e^{11}(k) & e^{12}(k) & \dots & e^{1(B_s+2)}(k)\\ e^{21}(k) & e^{22}(k) & \dots & e^{2(B_s+2)}(k)\\ \vdots & \vdots & & \vdots\\ e^{N1}(k) & e^{N2}(k) & \dots & e^{N(B_s+2)}(k)\end{bmatrix}.$$

In this error coordinate system, define the set

$$\begin{aligned}\mathcal{E}_0 &= \left\{E(k)\in\mathcal{E} : e_l^{ij} = 0,\ i\in\mathcal{N}, j\in\mathcal{B}, 1\le l\le n\right\}\\ &= \left\{\mathbf{0}^{Nn\times(B_s+2)}\right\}. \qquad (28)\end{aligned}$$

Let $\rho : \mathcal{E}\times\mathcal{E}\to\Re^+$ denote a metric on $\mathcal{E}$ and $\rho(E(k),\mathcal{E}_0)$ be the distance between $E(k)$ and the set $\mathcal{E}_0$. Since $\mathcal{E}_0$ has only one element, and it is the zero element, a valid choice for the metric is

$$\rho(E(k),\mathcal{E}_0) = \max_{1\le l\le n}\left\{\max_{i\in\mathcal{N}}\left\{\max_{j\in\mathcal{B}}\left\{\left|e_l^{ij}(k)\right|\right\}\right\}\right\}. \qquad (29)$$

For convenience, we collect some relevant parameters from Lemmas 4–6.

(Lemma 4) $\alpha = c_1^{(P+1)(B_s+1)+1}$, $\beta_1 = (1 - k_vNT - k_dT)(P+1)(B_s+1) - 2 + k_dT < 0$, and $\gamma = [1-(2-k_dT)^{(P+1)(B_s+1)+1}]/(k_dT-1)$, with $\gamma > 1$.

(Lemma 5) $\gamma_p = c_3^p\beta_1\gamma + \sum_{z=0}^{p-1} c_3^z\beta_2$ for $p\in\{0,1,\dots,P\}$ (and, particularly, $\gamma_0 = \beta_1\gamma$), with $\beta_2 = -(1-k_vT)(P+1)(B_s+1) - 2 + k_dT < 0$.

(Lemma 6) $\beta = \max\{-\gamma_0, -\gamma_P\} > 0$.

Next, we present our main results on uniform ultimate boundedness and exponential stability characterizations of cohesiveness. We show this via a general Lyapunov approach like that in [24].

Theorem 1: Given a multiagent system described by (10), if Assumptions 1 and 2 hold, then the trajectories of (10) are uniformly ultimately bounded in the error coordinate system. Let

$$\eta = 2\left(\frac{\beta}{c_3^P\alpha - \epsilon} + c_0\right)$$

where α is defined in Lemma 4, β is defined in Lemma 6, ϵ is any constant such that $0 < \epsilon < c_3^P\alpha < 1$, and $c_0 = (P+1)(B_s+1)+1$. Let $W(k) = M(k) - m(k)$ and $l'(k) = \arg\max_{1\le l\le n}\{W_l(k)\}$. Then, there exists a finite $k^0\in\mathcal{Z}^+$ such that

$$W_{l'(k)}(k) = M_{l'(k)}(k) - m_{l'(k)}(k) < \eta\Theta_0T^2 + \theta \qquad (30)$$

for all $k\ge k^0$, with $\theta > 0$ any positive real number. In particular, $k^0 = \max_{1\le l\le n}\{k_l^0\}$, where

$$k_l^0 = \begin{cases} 0, & \text{if } W_l(0) < c_\epsilon + \theta\\ \left\lceil \log_{1-\epsilon}\left(\frac{c_\epsilon+\theta}{W_l(0)}\right)\right\rceil c_0, & \text{otherwise}\end{cases}$$

with $c_\epsilon = 2\beta\Theta_0T^2/(c_3^P\alpha - \epsilon)$. Also, $|v^i(k)| < \eta\bar{\Theta}T + \theta/T$ for all $k\ge k^0$.

Proof: We employ a Lyapunov stability theoretic approach to prove this theorem. Choose Lyapunov function $V(k) = W_{l'(k)}(k)$. Obviously, $l'(k)$ is time-varying, which is allowed in the following analysis. In the error coordinate system, we have

$$\begin{aligned}W(k) &= \max_{i,j}\left\{x^{ij}(k) - \bar{x}(k)\right\} - \min_{i,j}\left\{x^{ij}(k) - \bar{x}(k)\right\}\\ &= \max_{i,j}\left\{e^{ij}(k)\right\} - \min_{i,j}\left\{e^{ij}(k)\right\}\end{aligned}$$

and, thus

$$\begin{aligned}V(k) &= \max_l\left\{\max_{i,j}\left\{e_l^{ij}(k)\right\} - \min_{i,j}\left\{e_l^{ij}(k)\right\}\right\}\\ &= \max_{i,j}\left\{e_{l'(k)}^{ij}(k)\right\} - \min_{i,j}\left\{e_{l'(k)}^{ij}(k)\right\}.\end{aligned}$$

Next, we show $V(k)$ is bounded from above and below by two $\mathcal{KR}$ functions.[1] Since

$$\max_{i,j}\left\{\left|e^{ij}(k)\right|\right\} \geq \max_{i,j}\left\{x^{ij}(k)-\bar{x}(k)\right\} = M(k)-\bar{x}(k)$$
$$\max_{i,j}\left\{\left|e^{ij}(k)\right|\right\} \geq \max_{i,j}\left\{\bar{x}(k)-x^{ij}(k)\right\} = \bar{x}(k)-m(k)$$

with $|\cdot|$ a componentwise absolute value, we have $M(k)-m(k) \leq 2\max_{i,j}\left\{\left|e^{ij}(k)\right|\right\}$. Then, by the definition of $V(k)$ and (29)

$$\begin{aligned} V(k) &= M_{l'(k)}(k) - m_{l'(k)}(k) \\ &\leq 2\max_{i,j}\left\{\left|e^{ij}_{l'(k)}(k)\right|\right\} \\ &\leq 2\max_{i,j,l}\left\{\left|e^{ij}_{l}(k)\right|\right\} \\ &= 2\rho\left(E(k),\mathcal{E}_0\right). \end{aligned}$$

Also, by definition, $\rho(E(k),\mathcal{E}_0) \leq M_{l'(k)}(k) - m_{l'(k)}(k)$. In summary, we have

$$\rho(E(k),\mathcal{E}_0) \leq V(k) \leq 2\rho(E(k),\mathcal{E}_0). \quad (31)$$

Next, we will show that for arbitrary l, $1 \leq l \leq n$, there exists a finite $k_l^0 \in \mathcal{Z}^+$ such that $W_l(k) < \eta\Theta_0 T^2 + \theta$ for all $k \geq k_l^0$. To do this, we first show that there must exist a finite k_l^0 such that $W_l(k_l^0) < c_\epsilon + \theta$, with $c_\epsilon = 2\beta\Theta_0 T^2/(c_3^P\alpha - \epsilon)$ as defined earlier. Obviously, $c_\epsilon \leq \eta\bar{\Theta}T^2$. Since $c_0 = (P+1)(B_s+1)+1$, from Lemma 6, we have for all $k \geq 0$,

$$W(k+c_0) \leq (1-c_3^P\alpha)W(k) + 2\beta\bar{\Theta}T^2.$$

Notice that $\bar{\Theta} = \Theta_0\mathbf{1}_n$, we have for each l

$$W_l(k+c_0) \leq (1-c_3^P\alpha)W_l(k) + 2\beta\Theta_0 T^2. \quad (32)$$

If $W_l(k) \geq c_\epsilon$, we have from (32) that

$$W_l(k+c_0) \leq (1-\epsilon)W_l(k). \quad (33)$$

Since $0 < \epsilon < c_3^P\alpha < 1$, given arbitrary $\theta > 0$, there must exist a finite $k_l^0 \in \mathcal{Z}^+$ such that $W_l(k_l^0) < c_\epsilon + \theta$. In particular, if $W_l(0) < c_\epsilon + \theta$, then $k_l^0 = 0$; otherwise, $W_l(0) \geq c_\epsilon + \theta$ and there exists an integer k_l such that $W_l(0)(1-\epsilon)^{k_l} < c_\epsilon + \theta$, where

$$k_l = \left\lceil \log_{1-\epsilon}\left(\frac{c_\epsilon+\theta}{W_l(0)}\right)\right\rceil \quad (34)$$

and, thus, $k_l^0 = k_l c_0$.

Next, we prove by induction that $W_l(k) < \eta\Theta_0T^2 + \theta$ for all intervals of $[k_l^0 + sc_0, k_l^0 + (s+1)c_0]$, $s \in \mathcal{Z}^+$. When $s = 0$, from Lemma 3 we have for all $k \in [k_l^0, k_l^0 + c_0]$

$$W_l(k) \leq W_l(k_l^0) + 2c_0\Theta_0T^2 < \eta\Theta_0T^2 + \theta$$

since $W_l(k_l^0) < c_\epsilon + \theta$. Suppose it holds for $s = s'$. Then for each $k \in [k_l^0 + s'c_0, k_l^0 + (s'+1)c_0]$, it must satisfy either *i)* $c_\epsilon \leq W_l(k) < \eta\Theta_0T^2 + \theta$ or, *ii)* $0 \leq W_l(k) < c_\epsilon$. If k is such that case *i)* holds, then we have $W_l(k+c_0) \leq (1-\epsilon)W_l(k)$ from (33); if k is such that case *ii)* holds, then we have, from Lemma 3, $W_l(k+c_0) \leq W_l(k) + 2c_0\Theta_0T^2$. In either case, $W_l(k+c_0) < \eta\Theta_0T^2+\theta, k \in [k_l^0+s'c_0, k_l^0+(s'+1)c_0]$. That is, $W_l(k) < \eta\Theta_0T^2+\theta$ for all $k \in [k_l^0+(s'+1)c_0, k_l^0+(s'+2)c_0]$.

Finally, since l is picked arbitrarily, we have $V(k) = W_{l'(k)}(k) = \max_l W_l(k) < \eta\Theta_0T^2 + \theta$ for all $k \geq k^0 = \max_{1\leq l\leq n}\{k_l^0\}$. The UUB property for $v^i(k)$ follows similarly. ■

For some applications, it is possible that the terms related to noise, repulsion and profile are negligible. That is, $\bar{\Theta} = 0$. So, (10) can be rewritten as

$$x^i(k+1) = c_1^i x^i(k) + c_2^i x^i(k-1) + c_3\sum_{j\in\Pi^i} x^j(k-\tau_j^i) + c_4\sum_{j\in\Pi^i} x^j(k-1-\tau_j^i). \quad (35)$$

Then, the stability property of this system is given by the following theorem.

Theorem 2: Given a multiagent system described by (35), if Assumptions 1 and 2 hold, then the set $\mathcal{E}_0$, defined in (28), is invariant and exponentially stable. Moreover, $v^i(k) \to 0$ for all i as $k \to \infty$.

Proof: To see why $\mathcal{E}_0$ is invariant, note that in $\mathcal{E}_0$ we have $e_l^{ij} = 0$ for all i, j, and l. Correspondingly, $x^{i_1j_1}(k) = x^{i_2j_2}(k) = \bar{x}(k)$ for all $i_1, i_2 \in \mathcal{N}$ and $j_1, j_2 \in \mathcal{B}$, with $x^{ij}(k)$ defined in (11). Recall that $c_1^i + c_2^i + c_3N^i + c_4N^i = 1$, so we have from (35)

$$x^i(k+1) = c_1^i\bar{x}(k) + c_2^i\bar{x}(k) + c_3\sum_{j\in\Pi^i}\bar{x}(k) + c_4\sum_{j\in\Pi^i}\bar{x}(k) = \bar{x}(k)$$

for all i at any time step $k \in \mathcal{Z}^+$. Thus, $x^i(k+1)$ is such that all $e_l^{ij}(k+1)$ are still in $\mathcal{E}_0$. So, $\mathcal{E}_0$ is invariant.

Before we proceed, we define a Lyapunov function in the error coordinate system. As in Theorem 1, let $l'(k) = \arg\max_{1\leq l\leq n}\{M_l(k) - m_l(k)\}$. Let $W(k) = M(k) - m(k)$. Choose Lyapunov function $V(k) = W_{l'(k)}(k)$. Note that by definition, if $k' \neq k$, then $W_{l'(k)}(k) \geq W_{l'(k')}(k)$. That is, at time step k, W_l gives the maximum when $l = l'(k)$, though possibly W_l reaches the maximum with some $l = l'(k') \neq l'(k)$ at some k' other than k.

To prove that $\mathcal{E}_0$ is exponentially stable, note that (31) holds here and we first show that $V(k)$ is nonincreasing in time. Notice when $\bar{\Theta} = 0$, Lemma 3 gives $M(k+1) \leq M(k)$ and $m(k+1) \geq m(k)$ and thus, $W_l(k+1) \leq W_l(k)$ for all l. Therefore,

$$V(k+1) = W_{l'(k+1)}(k+1) \leq W_{l'(k+1)}(k) \leq W_{l'(k)}(k) = V(k).$$

This means $V(k)$ is nonincreasing.

Next, we will show that there exist some constants c_v and α_v such that $\rho(E(k),\mathcal{E}_0) \leq c_v\exp(-\alpha_v k)\rho(E(0),\mathcal{E}_0)$, which means $\mathcal{E}_0$ is exponentially stable. When $\bar{\Theta} = 0$, (32) gives $W_l(k+c_0) \leq (1-c_3^P\alpha)W_l(k)$ for all l and all $k \in \mathcal{Z}^+$. So, $W_l(c_0) \leq (1-c_3^P\alpha)W_l(0)$ and by induction, we have for any

[1]Consider a continuous function $f: \Re^+ \to \Re^+$. If $f(0) = 0$, f is strictly increasing on $[0,\infty)$, and $\lim_{r\to\infty} f(r) = \infty$, then f is said to belong to class $\mathcal{KR}$[24].

$s \in \mathcal{Z}^+$ that $W_l(sc_0) \leq (1 - c_3^P\alpha)^s W_l(0)$. Recall that $W_l(k)$ is nonincreasing, so for any $k \in [sc_0, (s+1)c_0]$, and for any $s \in \mathcal{Z}^+$, we have

$$\begin{aligned} W_l(k) &\leq W_l(sc_0) \leq (1 - c_3^P\alpha)^s W_l(0) \\ &\leq (1 - c_3^P\alpha)^{(k/c_0)-1} W_l(0). \end{aligned}$$

That is, $W_l(k) \leq c_v \exp(-\alpha_v k) W_l(0)/2$, where $c_v = 2(1 - c_3^P\alpha)^{-1}$, and $\alpha_v > 0$ is such that $\exp(-\alpha_v) = (1 - c_3^P\alpha)^{1/c_0}$. Such choice of α_v is feasible since $1 - c_3^P\alpha \in (0,1)$. Since l is picked arbitrarily, by the definition of $V(k)$, we have

$$\begin{aligned} V(k) &= W_{l'(k)}(k) \\ &\leq \frac{c_v}{2}\exp(-\alpha_v k) W_{l'(k)}(0) \\ &\leq \frac{c_v}{2}\exp(-\alpha_v k) W_{l'(0)}(0) \\ &= \frac{c_v}{2}\exp(-\alpha_v k) V(0). \end{aligned}$$

Finally, since $\rho(E(k), \mathcal{E}_0) \leq V(k) \leq 2\rho(E(k), \mathcal{E}_0)$, the previous equation gives

$$\rho(E(k), \mathcal{E}_0) \leq c_v \exp(-\alpha_v k) \rho(E(0), \mathcal{E}_0).$$

So far we have proven the set $\mathcal{E}_0$ is exponentially stable, which means as $k \to \infty$,

$$e_l^{ij}(k) = x_l^{ij}(k) - \bar{x}_l(k) = x_l^i(k-j+1)(k) - \bar{x}_l(k) \to 0 \quad (36)$$

for all $i \in \mathcal{N}$, $j \in \mathcal{B}$, and $1 \leq l \leq n$. From (35) and (36), $x^i(k+1) \to \bar{x}(k)$ for all i. Since (36) holds for all l, we have $x^i(k+1) - x^{ij}(k) \to 0$ (componentwise) for all $i \in \mathcal{N}$ and $j \in \mathcal{B}$ as $k \to \infty$. Thus, as $k \to \infty$, $v^i(k) = \big(x^i(k+1) - x^i(k)\big)/T \to 0$. ■

D. Discussion: Parameters, Sensing Topology, and Extensions

1) Effects of Parameters: Before we proceed, it should be noted that the uniform ultimate bound we obtain in (30) can be quite conservative since we have to take overbounds of many terms during the deduction. Therefore, one has to be careful not to be too ambitious in interpreting the results. Nevertheless, as we discuss next, the above results are still useful for providing insights into the effects of system parameters on dynamics and cohesiveness.

If Assumption 1 holds, we have $1 < k_d T < 2$ or alternatively, $1/k_d < T < 2/k_d$, meaning that when k_d is fixed, the sampling time T can be neither too large nor too small. Also, $c_3 \in (0,1)$ gives $k_v < 1/T < k_d$. Recall that k_v is the "velocity attraction gain" and k_d is the "velocity damping gain;" hence, this means that if the damping term dominates the velocity attraction effect this could help to achieve uniform ultimate boundedness.

The parameters k_r, r_s and k_f do not affect the boundedness (i.e., Lagrange stability) of the system trajectories. Neither do the noise bounds δ_p, δ_v and δ_f and the magnitude of resource profile gradient Λ_0. However, all these parameters do affect the size of the ultimate bound on the system trajectories (if it is bounded at all). In fact, increasing these parameters could increase the uniform ultimate bound $\eta\Theta_0 T^2 + \theta$ since Θ_0 is increased.

Note that with a repulsion term of the form defined in (2), collision avoidance is not guaranteed. But large k_r and r_s, meaning strong repulsion effects, may help reduce collisions between the agents. Moreover, as in [10], [29], and [30] it is possible to extend the results of this paper to consider a "hard repel" case by using a different form for the repel term. Also, note that each agent can only sense its neighbors' positions, and thus the repulsion term only takes effect in this "neighborhood." In other words, if agent $j \notin \Pi^i$, then agent i will not "repel" agent j even if they are close to each other.

The parameters B_s and P do not affect the boundedness of the system trajectories. However, they may affect the ultimate bound on the system trajectories in that they affect η. To see this, first note that

$$\begin{aligned} -\gamma_0 &= -\beta_1\gamma \\ &= ((-1 + k_v NT + k_d T)(c_0 - 1) + 2 - k_d T) \\ &\quad \times \frac{1 - (2 - k_d T)^{c_0}}{k_d T - 1} \\ &> 0 \end{aligned}$$

with c_0 as defined in Theorem 1. When B_s increases, c_0 increases and thus, $-\gamma_0$ increases. Recall that $\beta = \max\{-\gamma_0, -\gamma_P\}$, so β increases as B_s increases. As a result, η increases. This means that a large B_s could lead to large ultimate bound on the system trajectories.

Note that although P affects η in a similar way as B_s does, one should not jump too quickly to the conclusion that a large P increases the bound. This is because, as discussed earlier, we overbound many terms and this leads to conservativeness. Specifically, recall that by definition, large P may mean that there are fewer neighbors (small N^i) for an agent i, which further means that the repulsion effect, quantified by ψ^i, on agent i could be small since $\psi^i \leq (N^i - 1)\exp(-1/2)k_r r_s$ from (5). In other words, although large P could increase the trajectory bound by increasing η, it could also decrease the trajectory bound by decreasing the upper bound of the repulsion term. Thus, without knowing the specific topology, we cannot say too much about the effect of P on the uniform ultimate bound.

The values of B_s and P also affect the convergence speed of the system. To see this, we assume for simplicity that $\bar{\Theta}$ is so small that $2\beta\bar{\Theta}T^2$ is negligible. Then, (26) can be written as

$$\begin{aligned} M(n_0 + (P+1)(B_s+1)+1) - m(n_0 + (P+1)(B_s+1)+1) \\ < (1 - c_3^P\alpha)(M(n_0) - m(n_0)). \end{aligned}$$

Note that when c_1 and c_3 are fixed, both c_3^P and $\alpha = c_1^{(P+1)(B_s+1)+1}$ decrease as P and B_s increase, meaning the system trajectory convergence speed decreases.

Theorem 2 indicates that for the multiagent system described by (35) a certain set is exponentially stable in the error coordinate system despite the existence of sensing delays and topology. In particular, all agents will converge to one point and ultimately stop. The effects of P and B_s on convergence speed are the same as stated before.

Note that in Assumption 1, $c_3 > 0$ and $c_4 \geq 0$ are equivalent to $c_3' = k_v NT > 0$ and $c_4' = k_p NT^2 - k_v NT \geq 0$. Inspecting c_1, c_2, c_3' and c_4' shows that all k_p and k_v appear with "N," which means by choosing $k_p = k_p'/N$ and $k_v = k_v'/N$, Assumption 1 can be made free of N. In other words, consider Assumption 3.

Assumption 3: Let $c_1' = 2 - k_v'T - k_dT$, $c_2' = -1 - k_p'T^2 + k_v'T + k_dT$, $c_3' = k_v'T$, and $c_4' = k_p'T^2 - k_v'T$. The parameters k_p', k_v', k_d, and T are such that $c_1' > 0$, $c_2' > 0$, $c_3' > 0$ and $c_4' \geq 0$.

If Assumption 3 holds and if k_p and k_v are free to change by design, then Assumption 1 holds for any N ($N \geq 2$). This is because we can choose k_p and k_v such that the effect of N is counteracted and thus, satisfaction of Assumption 1 holds independent of N. Accordingly, the uniform ultimate boundedness of the system is independent of the number of agents N, presuming Assumption 2 still holds.

2) Effects of Sensing Topology and Relations to a Switching Topology: It should be noted that some types of classical network topologies can easily fit into our model. For example, when their graphs are directed, *i)* a *line* topology can be characterized by $N^i = 2$ (recall $i \in \Pi^i$) for all $i \in \mathcal{N}$, $P = N - 1$, and $\mathcal{D} = \{1\}$, presuming agent 1 is one end of the topology, *ii)* a *ring* topology can be characterized by $N^i = 2$ for all $i \in \mathcal{N}$, $P = N - 1$, and $\mathcal{D} = \mathcal{N}$, and *iii)* a *completely connected* topology can be characterized by $N^i = N$ for all $i \in \mathcal{N}$, $P = 1$, and $\mathcal{D} = \mathcal{N}$.

As defined in Section II-A, the sensing topology in our model is a fixed (time-invariant) topology coupled with sensing delays and sensing errors. Next, we relate our sensing topology to a switching topology by comparing it with that in [19], where the authors define a switching topology to study Vicsek's model [3].

First, there are similarities between both topologies. Specifically, in [19] the authors use some switching signal $\sigma(t)$ to characterize the switching topology and, for any agents i and j, agent i cannot obtain the latest information ("heading angle") of agent j when the interconnection is broken. In our sensing topology, the inclusion of time-varying sensing delays and sensing errors represents that for any agents i and j, but $j \in \Pi^i$ and $j \neq i$, there is no guarantee that agent i obtains the latest information (position and velocity) of agent j accurately. So, this captures some features observed in a switching topology. Also, to achieve convergence, it is assumed in [19] that the switching signal $\sigma(t)$ is such that the agents are "linked together" across contiguous time intervals of *arbitrary but finite* length. Basically this means the information ("heading angle") of any agent j can affect, through certain path, any agent i in a finite-length time interval. In other words, j can affect i sufficiently frequently, although i and j possibly are not directly connected. Our results in earlier parts of Section III indicate that when some assumptions are satisfied, convergence can be achieved for *arbitrary but finite* B_s, which quantifies the largest amount of information outdate. Basically, this means it must take a finite-length time for the information (position and velocity) of j to affect i (suppose there exists a positive path from j to i) or, in other words, j can affect i sufficiently frequently, although i and j possibly are not directly connected.

There are also significant differences between our sensing topology and the one in [19]. First, the authors in [19] restrict to the topology to one characterized by some *undirected* graph, which means at any instant, if agent i can sense j, then j must be able to sense i. In our sensing topology, we allow the topology characterized by some *directed* graph, which means it is possible that agent i can sense j while j *cannot* sense i. Obviously, one can regard undirected graph as a special case of directed graph. However, to achieve this, we *require* the existence of a set of "distinguished" agents, i.e., some special type of agent. Next, for the switching topology in [19], at any time instant, agent i either senses the latest information of agent j or senses nothing about j. Our topology is different in that the sensing topology is a fixed topology, where at any time instant, agent i *always* has the potential to sense some position and velocity information about agent $j \in \Pi^i$, but such information could be always outdated (but the outdate amount is bounded). That is, it is possible that i *never* senses the latest information of j. Finally, by our definition of τ_j^i, or more explicitly, $\tau_j^i(k)$, it is possible that for $k' > k$, $k - \tau_j^i(k) > k' - \tau_j^i(k')$. (Of course, a necessary condition to have this happen is $k' - k < B_s$ since τ_j^i is bounded by B_s.) That is, suppose the position and velocity information about agent j are data indexed by k, then these data could arrive at agent i (suppose $j \in \Pi^i$) in a "shuffled" order. From our deduction, clearly this "order shuffling" phenomenon, with the constraint of $\tau_j^i \in [0, B_s]$, does not affect the uniform ultimate boundedness of the system. In comparison, the topology in [19] does not include a time delay, so it is not immediately clear whether "order shuffling" could be allowed in that framework.

3) Extensions: Here, we discuss some possible generalizations of our results. It is possible to generalize the form of control input, as given in (3), to accommodate a class of attraction functions that include a nonlinearity and obtain some results similar to those in [29]. In particular, consider the function $g : \Re^n \to \Re^n$, $g(x) = [g_1(x_1), g_2(x_2), \ldots, g_n(x_n)]^\top$ for $x \in \Re^n$. Suppose for all $1 \leq l \leq n$, the component function $g_l : \Re \to \Re$ is odd, continuous, and satisfies $\mu_1 \leq g_l(y)/y \leq \mu_2$ for all $y \in \Re$, $y \neq 0$, with μ_1 and μ_2 some known positive constants, and $g_l(0) = 0$. Then, the generalized control input can have the following form:

$$\begin{aligned} u^i(k) = &- M_i k_p \sum_{j \in \Pi^i} g\left(\hat{x}^i(k) - \hat{x}^j(k - \tau_j^i)\right) \\ &- M_i k_v \sum_{j \in \Pi^i} g\left(\hat{v}^i(k) - \hat{v}^j(k - \tau_j^i)\right) - M_i k_d \hat{v}^i(k) \\ &+ M_i k_r \sum_{j \in \Pi^i} \exp\left(\frac{-\frac{1}{2}\left\|\hat{x}^i(k) - \hat{x}^j(k - \tau_j^i)\right\|^2}{{r_s}^2}\right) \\ &\times \left(\hat{x}^i(k) - \hat{x}^j(k - \tau_j^i)\right) - M_i k_f \nabla \hat{J}^i(k). \end{aligned}$$

Obviously, when $\mu_1 = \mu_2 = 1$, the previous equation changes into (3).

Also, it is possible to cope with gradient following and trajectory following with our model. For gradient following, the agents only try to have the gradient of their position trajectories be the same as that of the desired trajectory, $J_d(k)$, and to achieve this, we just need to change $\nabla J^i(k)$ (where for simplicity, we ignore the noise effect) in (3) into $-\nabla J_d^i(k)$. For trajectory following, the agents try to have their position trajectories track the desired trajectory, and to achieve this, one possible way is to replace $\nabla J^i(k)$ in (3) with $x^i(k) - J_d^i(k)$. As one can imagine, generally, the agent position trajectories generated by gradient following and trajectory following will both have the same *shape* as $J_d(k)$, though possibly are scaled by some factor. However, gradient following also allows some offset (or translation) on the agent position trajectories with respect to $J_d(k)$, i.e., in *location* the position trajectories of all the agents can be quite different from $J_d(k)$, though they have the same *orientation*. In comparison, trajectory following guarantees that the agent position trajectories and $J_d(k)$ are at the same location. It is clear that gradient following can be accommodated in our model without any change, and all our previous proofs hold. While for trajectory following, some changes in the control input (3) and thus, system (10), are required, and the proofs would need to change. Nevertheless, it should be noted that with either gradient following or trajectory following, the resultant multiagent model is closely related to models of *coupled synchronization*, a phenomenon ubiquitous in nature and one that is attracting increasing research interest [31]–[33]. In Section IV-B, we will show some simulation results which will help to make connections between multiagent system cohesion and synchronization.

IV. Applications

The swarm models given in [12], [21] and this paper can have different applications. Some candidates, as indicated in [12] and [21], include groups of robots designed to coordinate their activities, networked cooperative UAVs developed for commercial and military purposes, platooning of vehicles in IVHS, and so on. Also, from the theory in this paper, we can see that the idea of the agreement problem [23] is also reflected in the models. Moreover, as we stated in Section III-D3, the models are also related to models of synchronization. Next, we will show in Section IV-A some simulations where each agent is regarded as a vehicle with second-order dynamics. The simulation results verify some of our earlier observations in Section III. Then, in Section IV-B, we will give some simulation results on synchronization, with gradient/trajectory following included in the model, which we hope will motivate future research in understanding the relations between multiagent systems and coupled oscillators.

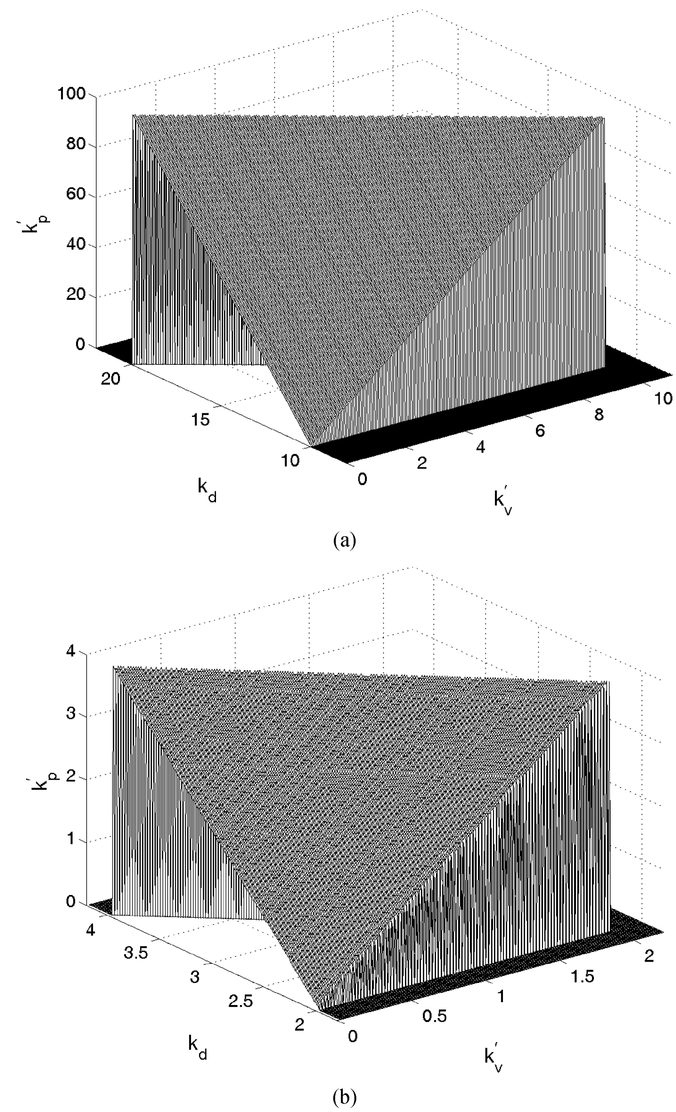


Fig. 2. Parameter surface qualified for Assumption 3. (a) $T = 0.1$. (b) $T = 0.5$.

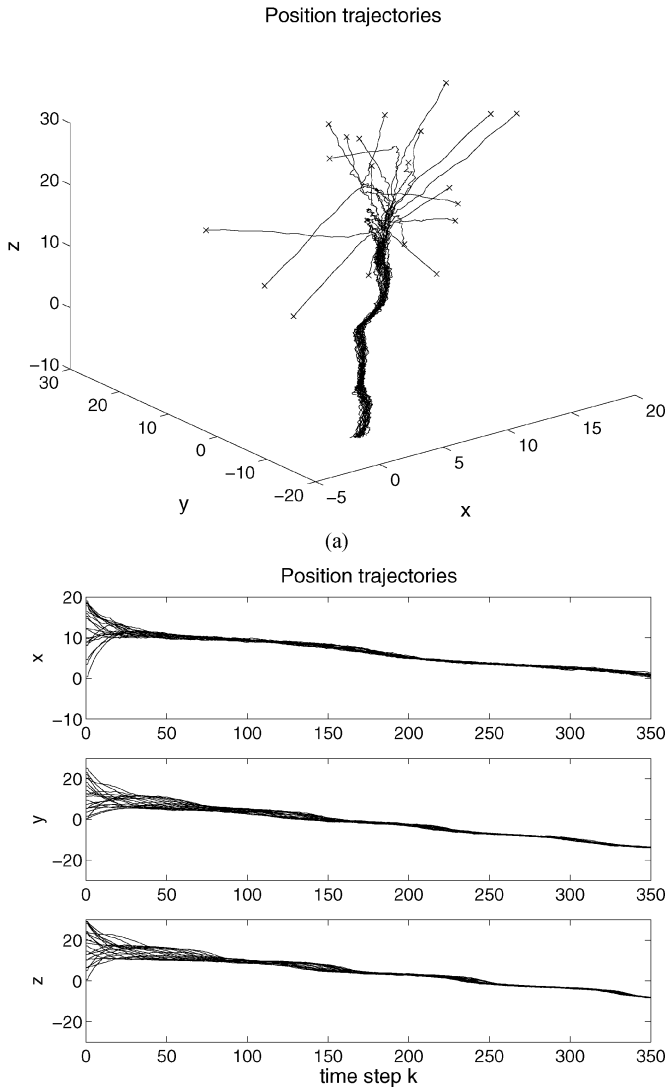
A. Multivehicle Cohesion

In this section, we will show some simulation results where we view the agents as vehicles. First, we show some plots related to the parameter triplet (k_p', k_v', k_d) in Assumption 3. It can be shown that such a parameter set is nonempty for any $T > 0$. For comparison, we arbitrarily pick $T = 0.1$ and $T = 0.5$. The corresponding parameter surfaces are shown in Fig. 2, with each point on the surface corresponding to a triplet. For a triplet, if none of the three components is zero, then it satisfies Assumption 3. Consider the $T = 0.1$ case in Fig. 2(a). Obviously, k_d has its valid range in (10, 20), which reflects our earlier observation of $1/T < k_d < 2/T$. Similarly, $k_p' \in (0, 1/T^2) = (0, 100)$ since $c_1' + c_2' > 0$, and $k_v' \in (0, 1/T) = (0, 10)$ since $c_1' + c_2' + c_4' > 0$.

Before we proceed, we will specify how to construct a sensing topology with a given P since different P will be used in the following simulations. Recall that for agent i, Π^i is the set of its neighbors, and $i \in \Pi^i$. Suppose each agent i has (including itself) c_P neighbors in a "circulant" manner, that is, $\Pi^1 = \{1, 2, \ldots, c_P\}$, $\Pi^2 = \{2, 3, \ldots, c_P + 1\}, \ldots$, and $\Pi^N = \{N, 1, \ldots, c_P - 1\}$. Obviously, we have $\mathcal{D} = \mathcal{N}$ in this case. Choose $c_P = \lceil N/P \rceil$, then the resultant sensing topology has the desired P. Of course, other approaches, including some that allow randomly assigned connections, to construct a sensing topology such that it has the desired P, are possible.

In all the following simulations, unless otherwise stated, the system parameters are $N = 20$, $k_p = 0.2$, $k_v = 0.06$, $k_d = 6$, $k_r = 1$, $r_s = 0.8$, $k_f = 1.1$, and $T = 0.2$. The noise bounds are $\delta_p = \delta_v = \delta_f = 2\sqrt{3}$, assuming that the noise d_p^{ij}, d_v^{ij} and d_f^i are uniformly distributed and zero-mean for all $i, j \in \mathcal{N}$. Also, the profile gradient $\nabla J = [\cos(x) + 2,\ 2\sin(y) + 4,\ 3\cos(z) + 4.5]^\top \in \Re^3$, $P = 4$, and sensing delay $B_s = 20$. All the simulations in this section are run for 350 time steps. All the agents are assigned initial positions randomly. For simplicity, their positions are kept constant for $k \leq 0$ and correspondingly, all agents have zero velocities for $k < 0$.

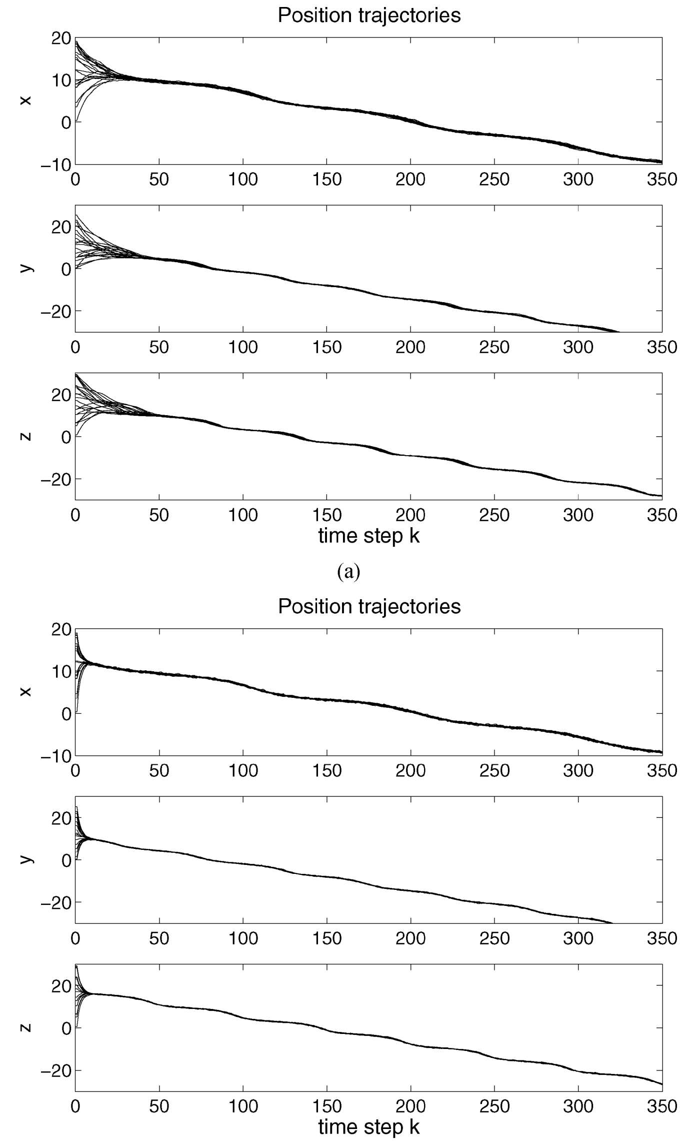
Fig. 3 shows the position and velocity trajectories of the system. From Fig. 3(a), we can see that at the beginning of the simulation, the agents appear to move around erratically.



(a)

(b)

Fig. 3. Position trajectories ((x, y, z) denotes the three dimensions). (a) Position trajectories in (x, y, z). (b) Position trajectories versus time.



(a)

(b)

Fig. 4. Position trajectories with parameter changes. (a) No sensing delay case ($B_s = 0$). (b) Complete sensing graph ($B_s = 0$ and $P = 1$).

Soon, they get close to each other and move cohesively in spite of the existence of noise, sensing delays, and topology characterized by an incompletely connected graph. Fig. 3(b) shows the position trajectories in three dimensions versus time.

Next, we illustrate the effects of P and B_s. Fig. 4(a) is for the case of no sensing delay ($B_s = 0$), and Fig. 4(b) is for the case where we further have a completely connected sensing topology, i.e., $B_s = 0$ and $P = 1$. As expected, comparing Fig. 3(b) with Fig. 4(a) and (b), we can see that decreasing B_s and P helps increase convergence speed. Effects of other parameters are also as expected.

B. Synchronization of Coupled Oscillators

In this part, we show an example where the agents achieve a type of synchronization through gradient following or trajectory following, as discussed in Section III-D3, and slide down a spiral line in a cohesive manner. The three-dimensional spiral line, which is the desired trajectory, is defined as $J_d(k) = [\cos(z(k)), \sin(z(k)), z(k)]^\top \subset \Re^3$, with $z(k) = -R_z kT$, and R_z some known positive constant. Projected onto the (x, y) plane, this is a unit circle, with the origin its center, and we think of each agent's position in (x, y) as representing the phase of an oscillator. To illustrate the key idea of synchronization of coupled oscillators, we remove sensing errors and interagent repulsion from the model. So, $k_r = \delta_p = \delta_v = \delta_f = 0$. Also, we let $R_z = 0.1$, $B_s = 10$, and $k_f = 8.5$ for gradient following, and $k_f = 2$ for trajectory following. Other parameters, and the sensing topology, are as defined in Section IV-A. Simulations in this section are run for 1000 steps. The simulation results for gradient following are shown in the following figures, where the initial and final positions of an agent are represented by a cross sign and a black dot, respectively. (Note that only one black dot is shown in the figures since the agents are all on top of one another.) For comparison, the plot of $J_d(k)$ is superimposed in the figures as a dashed line with square markers.

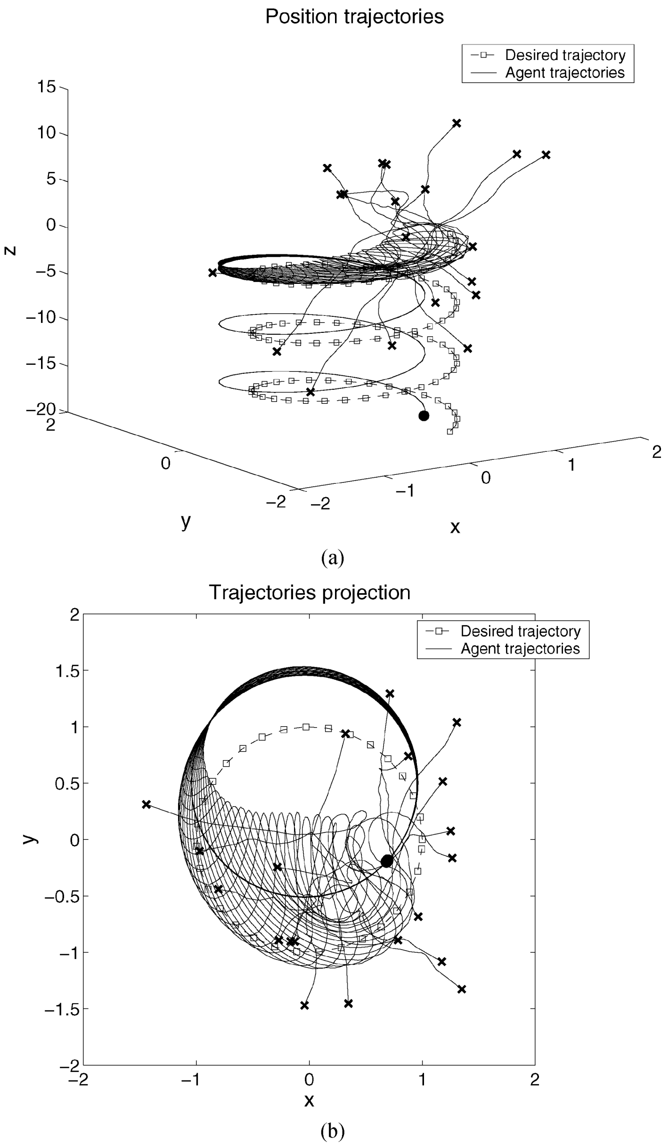


Fig. 5. Position trajectories with gradient following ((x, y, z) denotes the three dimensions). (a) Position trajectories in (x, y, z). (b) Position trajectories projection to (x, y).

Fig. 5(a) shows that, although their initial positions and velocities are assigned randomly, the agents soon move together, achieve synchronization, and slide down along a spiral line as a group. Since gradient following does not enforce alignment of the agent trajectories and the desired trajectory, the projection of the agent position trajectories on (x, y) plane has an offset on its center with respect to the one formed by $J_d(k)$, as shown in Fig. 5(b). The results for trajectory following (not shown here) is similar to those for gradient following, except that, as expected, the agent position trajectories and $J_d(k)$ are aligned with each other, i.e., their projections onto (x, y) plane are concentric circles. Overall, the simulations clearly illustrate the complicated nature of the dynamics of the distributed systems we are studying.

It should be noted that the synchronization shown by the above simulations is different from that found in [31], [33] since *i*) the agent trajectories here initially do not have to stay on a circle in the (x, y) plane, even though ultimately they must, in order to be synchronized, and *ii*) the underlying dynamics are different. Still, there are interesting analogies between the attraction terms and the sinusoidal terms in the Kuramoto model [31], and the velocity damping term along with the desired trajectory profile and the natural frequencies of the oscillators in [31].

V. Concluding Remarks

In this paper, we focused on an asynchronous discrete-time formulation and, using a Lyapunov approach, derived stability conditions under which a multiagent system achieves cohesiveness even in the presence of sensing delays, sensing errors, and sensing topology, though collision avoidance is not guaranteed in this scheme. An agent sensing topology characterized by a completely connected *undirected* graph might be easy to analyze, but may rarely be found in nature. By defining a directed graph $\mathcal{G}$ as in Section II-A, we remove the requirement (as in [12]) that each agent has to be able to detect all other agents. Although Assumption 2 requires the existence of some special set $\mathcal{D}$, but it is not a strong assumption. Thus, the results in this paper represent our progress in studying multiagent systems that demonstrate certain global "emergent" behaviors through local interactions. As discussed in Section III-D, our sensing topology with time-varying sensing delays and sensing errors captures some features observed in a switching (or time-varying) topology. Thus, we view the results in this paper as representing some progress toward establishment of stability properties of multiagent systems with a switching topology.

Finally, as explained in Section III-D3, our approach may also help make progress in studying the problem of synchronization of coupled oscillators with a sensing topology, time delays, and asynchronism. In fact, we hope that the ideas there will motivate the future study of relationships between multiagent system stability and synchronization.

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