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Decentralized output regulation of a class of nonlinear systems

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In this paper we investigate the decentralized output regulation problem of a class of nonlinear systems. It is shown that the results of decentralized output regulation of linear systems can be easily adapted to nonlinear systems within the Isidori-Byrnes framework. The resulting decentralized controller consists of local controllers, each of which is a parallel connection of a stabilizer and a (partial) internal model. We present only local results.

1. Introduction

The output regulation problem, or the so-called servomechanism problem, has been studied extensively in the past few decades. For the class of linear systems, the problem was studied and solved in the 1970's. See for example Davison and Goldenberg (1975), Davison (1976), Francis and Wonham (1976), and Francis (1977). The output regulation of nonlinear systems was first pursued by Huang and Rugh (1990) for systems with constant exogenous signals and by Isidori and Byrnes (1990) for more general class of exosystems (see also Isidori (1995)). In Huang and Rugh (1992a) the servomechanism problem for systems with slowly varying but not necessarily bounded exogenous signals was addressed, and a solution method based on the series expansion of the system functions and the solution of the regulator equations was presented. It was also shown that the solution of the problem depends also on the higher-order harmonics of the system. Later Huang and Rugh (1992b) extended the results to present an approximate method for calculating the solution of the regulator equations and showed that under the developed strategy a "guaranteed" bounded tracking is achieved where the bound on the tracking error depends on the quality of the approximation. Similarly, in Chu and Huang (1999), Wang *et al*. (2000) neural networks were used to approximate the solutions of the regulator equations. Recent research in this area has been focusing on robust regional, semiglobal, or global regulation of nonlinear systems. See for example Khalil (2000), Serrani *et al*. (2000), Serrani and Isidori (2000), and Serrani *et al*. (2001). In Serrani *et al*. (2001) the author used adaptive internal model for semiglobal output regulation in presence of unknown (but parameterized) linear exosystem.

The decentralized servomechanism problem for linear systems was considered by Davison (1976), where he provides necessary and sufficient conditions for the solvability of the problem. Here, we extend his results for the regulation of nonlinear systems in the framework of Isidori and Byrnes (1990) (see also Isidori (1995)). An initial version of this article was published in Gazi and Passino (2001). Later, in Ye and Huang (2003) decentralized adaptive output regulation of large-scale systems composed of sub-systems of the form of those discussed in Serrani and Isidori (2000), but with adaptive internal model as in Serrani *et al*. (2001) were considered. Note that our results are different from those in Ye and Huang (2003) and were obtained independently

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(and before) of the results in Ye and Huang (2003). In particular, they consider a class of interconnected systems in which each subsystem is minimum phase and has constant relative degree, whereas no such assumptions are made here. On the other hand, in Ye and Huang (2003) the authors use adaptive internal models and obtain global results, whereas we do not use parameter adaptation and our results are local.

Decentralized systems and in particular interconnected systems may arise in a variety of different applications. One possible application is the formation control (or in more general sense coordination and control) of multi-agent systems where each agent has its own vehicle dynamics; however, their motion is mutually constrained due to the coordination strategy. Preliminary work on application of the decentralized output regulation concepts to the formation control problem can be found in Gazi (2005).

2. The decentralized output regulation problem

In this section we consider the problem of finding a decentralized controller for the output regulation of a class of decentralized nonlinear systems of the form shown in figure 1. In other words, we consider systems which have ν local input and output terminals and are described by

$$\begin{aligned} \dot{x} &= f(x, w, u_1, \ldots, u_\nu), \\ e_i &= h_i(x, w), 1 \le i \le \nu, \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u_i \in \mathbb{R}^{m_i}$ and $e_i \in \mathbb{R}^{m_i}$, $1 \le i \le \nu$, $(m = \sum_{i=1}^{\nu} m_i)$ are the control inputs and outputs at each local station, respectively. The functions f and h_i, $i = 1, \ldots, \nu$, are known and smooth, and m_i, $i = 1, \ldots, \nu$, and ν are known. The signal $w \in \mathbb{R}^r$ represents the exogenous inputs, that are the reference inputs, that need to be tracked, and the disturbances, that need to be rejected. It is assumed that all the exogenous signals are generated by a neutrally stable exosystem

$$\dot{w} = s(w), \tag{2}$$

where the function s is known and smooth.

The problem is to regulate each of the local outputs e_i, $i = 1 \ldots \nu$, to zero using decentralized controls u_i, $i = 1 \ldots \nu$, that use only local error feedback. In other words, the control inputs u_i use the information only from their corresponding (local) outputs e_i as can be seen from figure 1. We define the problem as follows.

Decentralized Output Regulation Problem (DORP): Given a nonlinear system of the form of (1) and a neutrally stable exosystem in the form of (2), find, if possible, ν integers $p_1, p_2 \ldots, p_\nu$, and mappings $\eta_i(\xi_i, e_i)$ and $\theta_i(\xi_i)$, $1 \le i \le \nu$, where $\xi_i \in \mathbb{R}^{p_i}$, such that the following conditions are satisfied:

(S) The equilibrium $(x, \xi_1, \ldots, \xi_\nu) = (0, 0, \ldots, 0)$ of

$$\begin{aligned} \dot{x} &= f(x, 0, \theta_1(\xi_1), \ldots, \theta_\nu(\xi_\nu)), \\ \dot{\xi}_i &= \eta_i(\xi_i, h_i(x, 0)), \quad 1 \le i \le \nu, \end{aligned} \tag{3}$$

is locally exponentially stable.

(R) There exists a neighborhood V of the origin of $X \times \Omega \times W$, where $\Omega = \Omega_1 \times \cdots \times \Omega_\nu$, with $\Omega_i \subset \mathbb{R}^{p_i}$, such that, for each initial condition $(x(0), \xi(0), w(0)) \in V$ (where $\xi(0) = [\xi_1^\top(0), \ldots, \xi_\nu^\top(0)]^\top$), the solution of the system

$$\begin{aligned} \dot{x} &= f(x, w, \theta_1(\xi_1), \ldots, \theta_\nu(\xi_\nu)), \\ \dot{\xi}_i &= \eta_i(\xi_i, h_i(x, w)), \quad 1 \le i \le \nu, \\ \dot{w} &= s(w) \end{aligned} \tag{4}$$

satisfies the condition

$$\lim_{t \to \infty} h_i(x(t), w(t)) = 0, \tag{5}$$

for all $1 \le i \le \nu$.

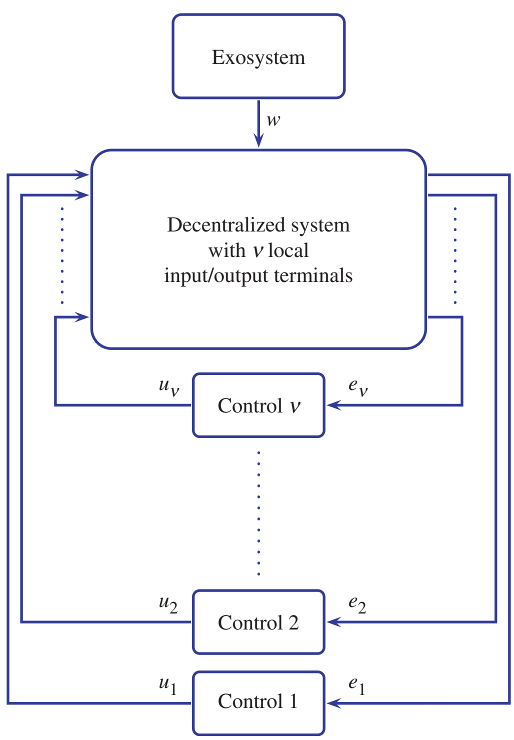

Figure 1. Decentralized system with ν input/output terminals.

Above the condition **(S)** stands for stability and the condition **(R)** stands for regulation. In the following analysis we use the linearization of the system around the operating point (the origin). Therefore, before proceeding we make the following definitions. Let

$$A = \frac{\partial f}{\partial x}(0,0,0), B_i = \frac{\partial f}{\partial u_i}(0,0,0), C_i = \frac{\partial h_i}{\partial x}(0,0),$$

$$S = \frac{\partial s}{\partial w}(0), F_i = \frac{\partial \eta_i}{\partial \xi_i}(0,0), G_i = \frac{\partial \eta_i}{\partial e_i}(0,0),$$

and

$$H_i = \frac{\partial \theta_i}{\partial \xi_i}(0).$$

Then, using these define $B = [B_1, \ldots, B_\nu]$, $C = [C_1^\top, \ldots, C_\nu^\top]^\top$, $F = bd[F_1, \ldots, F_\nu], G = bd[G_1, \ldots, G_\nu]$, and $H = bd[H_1, \ldots, H_\nu]$, where *bd* stands for *block diagonal*. Now, we have the following result, which is a decentralized version of those in Isidori and Byrnes (1990).

Lemma 1: Assume that for some $\eta_i(\xi_i, e_i)$ and $\theta_i(\xi_i)$, $1 \le i \le \nu$, the condition **(S)** is satisfied. Then, the condition **(R)** is also satisfied if, and only if, there exist mappings $x = \pi(w)$ and $\xi_i = \sigma_i(w)$, $1 \le i \le \nu$, with $\pi(0) = 0$ and $\sigma_i(0) = 0$, $1 \le i \le \nu$, defined in a neighborhood W^o of the origin of $\mathbb{R}^r$ satisfying the conditions

$$\begin{aligned} \frac{\partial \pi}{\partial w} s(w) &= f(\pi(w), w, \theta_1(\sigma_1(w)), \ldots, \theta_\nu(\sigma_\nu(w))), \\ \frac{\partial \sigma_i}{\partial w} s(w) &= \eta_i(\sigma_i(w), 0), \quad 1 \le i \le \nu, \\ 0 &= h_i(\pi(w), w), \quad 1 \le i \le \nu, \end{aligned} \tag{6}$$

for all $w \in W^o$.

Proof: *Necessity:* Since the system satisfies condition **(S)** with the above controller we have that the eigenvalues of the matrix (that is the linearization of the closed loop system around the origin)

$$\begin{bmatrix} A & BH \\ GC & F \end{bmatrix}$$

are located on the open left-half complex plane, whereas, the eigenvalues of S (the linearization of the exosystem) are all on the imaginary axis (because of its neutral stability). From the center manifold theory [20] we know that there exists a center manifold $x = \pi(w)$ and $\xi_i = \sigma_i(w)$, $1 \le i \le \nu$, such that the following equations are satisfied

$$\begin{aligned} \frac{\partial \pi}{\partial w} s(w) &= f(\pi(w), w, \theta_1(\sigma_1(w)), \ldots, \theta_\nu(\sigma_\nu(w))), \\ \frac{\partial \sigma_i}{\partial w} s(w) &= \eta_i(\sigma_i(w), h_i(\pi(w), w)), \quad 1 \le i \le \nu. \end{aligned} \tag{7}$$

Now, assume that the condition **(R)** is satisfied but the last equalities in (6) do not hold. Then, there is an output i, $1 \le i \le \nu$, such that for some w^o and $\pi(w^o)$ we have

$$\|h_i(\pi(w^o), w^o))\| = M_i > 0$$

and there exists a neighborhood U of $(\pi(w^o), w^o)$ such that

$$\|h_i(\pi(w), w))\| > M_i/2$$

for all $(\pi(w), w) \in U$. On the other hand, since condition **(R)** holds there exists a time $T > 0$ such that for all $t > T$

$$\|h_i(\pi(w(t)), w(t)))\| < M_i/2.$$

However, since the exosystem is neutrally stable, always there is some time $t_1 > T$ such that $(\pi(w(t_1)), w(t_1)) \in U$ which leads to a contradiction. Therefore, the last equalities in (6) hold. Substituting $h_i(\pi(w), w) = 0$ in (7) implies that all the equalities in (6) hold.

Sufficiency: Assume that (6) are satisfied. Then, by construction $x = \pi(w)$ and $\xi_i = \sigma_i(w)$, $1 \le i \le \nu$, constitute a center manifold for the system. From the properties of the center manifolds we know that for some $M > 0$ and $a > 0$ we have

$$\|\bar{x}(t) - \bar{\pi}(w(t))\| \le Me^{-at}\|\bar{x}(0) - \bar{\pi}(w(0))\|, \quad \forall t \ge 0,$$

where $\bar{x} = [x, \xi_1, \ldots, \xi_\nu]^\top$ and $\bar{\pi} = [\pi, \sigma_1, \ldots, \sigma_\nu]^\top$. Define $\tilde{x} = x(t) - \pi(w(t))$. Then, since as $t \to \infty$ we have $\tilde{x} \to 0$ exponentially fast, we obtain

$$\lim_{t\to\infty} e_i(t) = \lim_{t\to\infty} h_i(\pi(w(t)) + \tilde{x}(t), w(t)) = h_i(\pi(w(t)), w(t)) = 0,$$

for all $1 \le i \le \nu$. Therefore, the condition **(R)** is satisfied, and this completes the proof. □

In order to be able to achieve output regulation using decentralized controller one needs to be able to achieve stabilization using decentralized controller. Moreover, in the problem of decentralized stabilization the issue of decentralized fixed modes plays an important role. Therefore, since we are concerned with the problem of decentralized output regulation in this paper, before proceeding further we have to discuss the notion of *fixed modes*, a concept introduced first by Davison (1976) and Wang and Davison (1973).

Consider a linear time invariant system described by the triple $(C, A, B) \in \mathbb{R}^{m\times n} \times \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$. Let $\mathbf{K}$ be a set of matrices in $\mathbb{R}^{m\times m}$. Then, the set of fixed modes of (C, A, B) with respect to the set $\mathbf{K}$ is defined as follows:

$$\Lambda(C, A, B, \mathbf{K}) = \bigcap_{K\in\mathbf{K}} \lambda(A + BKC),$$

where $\lambda(A + BKC)$ is the set of eigenvalues of $(A + BKC)$.

In other words, the set of fixed modes with respect to a given set of matrices are the set of eigenvalues that cannot be changed by an output feedback with a gain matrix from this set. Note that $\Lambda(C, A, B, \mathbb{R}^{m\times m})$ is the set of the modes of A that are either uncontrollable or unobservable. Due to the decentralized structure of our problem, we consider only the set of block diagonal gain matrices

$$\mathbf{K_{bd}} = \left\{K\colon K = bd[K_1, \ldots, K_\nu], K_i \in \mathbb{R}^{m_i\times m_i}, i = 1, \ldots, \nu\right\}.$$

Note that the fixed modes of the system with respect to $\mathbf{K_{bd}}$ are called the *decentralized fixed modes* of the system. The location of the decentralized fixed modes is crucial for the solvability of the decentralized stabilization problem and therefore also for the solvability of the DORP.

Now, we are ready to state the conditions for the solvability of the DORP.

Theorem 1: The DORP is solvable if, and only if, there exist mappings $x = \pi(w)$ and $u_i = c_i(w), 1 \le i \le \nu$, with $\pi(0) = 0$ and $c_i(0) = 0$, $1 \le i \le \nu$, all defined in a neighborhood W^o of the origin of $\mathbb{R}^r$ and satisfying the conditions

$$\begin{aligned}\frac{\partial\pi}{\partial w} s(w) &= f(\pi(w), w, c_1(w), \ldots, c_\nu(w)),\\ 0 &= h_i(\pi(w), w), \quad 1 \le i \le \nu,\end{aligned} \tag{8}$$

for all $w \in W^o$, and such that the autonomous systems

$$\dot{w} = s(w), \quad u_i = c_i(w), \quad 1 \le i \le \nu, \tag{9}$$

are immersed into

$$\begin{aligned}\dot{\xi}_i &= \varphi_i(\xi_i),\\ u_i &= \gamma_i(\xi_i), \quad 1 \le i \le \nu,\end{aligned} \tag{10}$$

defined on neighborhoods $\Omega_i, 1 \le i \le \nu$, of the origins of $\mathbb{R}^{p_i}$, respectively, in which $\varphi_i(0) = 0$ and $\gamma_i(0) = 0$, $1 \le i \le \nu$, and the matrices

$$\Phi_i = \left[\frac{\partial\varphi_i}{\partial\xi_i}\right]_{\xi_i=0} \quad \text{and} \quad \Gamma_i = \left[\frac{\partial\gamma_i}{\partial\xi_i}\right]_{\xi_i=0}$$

for $1 \le i \le \nu$ are such that all the fixed modes with respect to $\mathbf{K_{bd}}$ (the decentralized fixed modes) of the triple

$$\bar{C} = \begin{bmatrix} C & 0\end{bmatrix}, \ \bar{A} = \begin{bmatrix} A & B\Gamma\\ NC & \Phi\end{bmatrix}, \ \bar{B} = \begin{bmatrix} B\\ 0\end{bmatrix}, \tag{11}$$

where $\Phi = bd[\Phi_1, \ldots, \Phi_\nu]$ and $\Gamma = bd[\Gamma_1, \ldots, \Gamma_\nu]$, have negative real parts, for some choice of $N = bd[N_1, \ldots, N_\nu]$.

Proof: *Necessity*: Suppose that the local controllers

$$\begin{aligned}\dot{\xi}_i &= \eta_i(\xi_i, e_i),\\ u_i &= \theta_i(\xi_i), \quad 1 \le i \le \nu,\end{aligned}$$

solve the decentralized output regulation problem. Then, by lemma 1 there exist mappings $x = \pi(w)$ and $\xi_i = \sigma_i(w)$, $1 \le i \le \nu$, with $\pi(0) = 0$ and $\sigma_i(0) = 0$, $1 \le i \le \nu$, such that (6) are satisfied. Set $c_i(w) = \theta_i(\sigma_i(w))$, $\gamma_i(\xi_i) = \theta_i(\xi_i)$, $\varphi_i(\xi_i) = \eta(\xi_i, 0)$, $1 \le i \le \nu$. Now, note that these satisfy (8). Moreover, we have $(\partial\sigma_i/\partial w)s(w) = \varphi_i(\sigma_i(w))$ and $c_i(w) = \gamma_i(\sigma_i(w))$, $1 \le i \le \nu$, implying that (9) are immersed into (10) for all $1 \le i \le \nu$. Moreover, since the given local controllers solve the regulation problem, the eigenvalues of the matrix

$$\begin{bmatrix} A & B\Gamma\\ GC & \Phi\end{bmatrix}$$

are all located in the open left-half plane. This, on the other hand, implies that all the fixed modes of the triple $(\bar{C}, \bar{A}, \bar{B})$ in (11) have negative real parts for $N = G$, i.e., $N_i = G_i, 1 \le i \le \nu$.

Sufficiency: Choose N_i, $1 \le i \le \nu$, such that the triple $(\bar{C}, \bar{A}, \bar{B})$ in (11) has all of its fixed modes with negative real parts. Then from theorem 1 in Wang and Davison (1973) we know that the decentralized stabilization problem of the system described by the above triple is solvable using dynamic output feedback. In other words, there exist integers $q_1, \ldots, q_\nu$, all greater than or equal to zero, and real constant matrices of the form $M = bd[M_1, \ldots, M_\nu], L = bd[L_1, \ldots, L_\nu]$, and $\Psi = bd[\Psi_1, \ldots, \Psi_\nu]$, where $L_i \in \mathbb{R}^{q_i\times m_i}, M_i \in \mathbb{R}^{m_i\times q_i}$, and $\Psi_i \in \mathbb{R}^{q_i\times q_i}$, such that the roots of the polynomial

$\det(\lambda I - A_e - B_e K_e C_e)$ have negative real parts. Above the matrices K_e, A_e, B_e, and C_e are

$$\begin{bmatrix} 0 & M \\ L & \Psi \end{bmatrix}, \begin{bmatrix} \bar{A} & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \bar{B} & 0 \\ 0 & I \end{bmatrix}, \begin{bmatrix} \bar{C} & 0 \\ 0 & I \end{bmatrix},$$

respectively, and $\bar{C}$, $\bar{A}$, and $\bar{B}$ are the matrices in (11).

Then choose the dynamics of each of the stabilizing local compensators as

$$\dot{\chi}_i = \Psi_i \chi_i + L_i e_i, \ 1 \le i \le \nu, \tag{12}$$

and the overall control input as

$$u_i = \gamma_i(\xi_i) + M_i \chi_i, \tag{13}$$

which render the matrix $(A_e + B_e K_e C_e) =$

$$\begin{bmatrix} A & B_1\Gamma_1 & B_2\Gamma_2 & \dots & B_\nu\Gamma_\nu & B_1M_1 & \dots & B_\nu M_\nu \\ N_1C_1 & \Phi_1 & 0 & \dots & 0 & 0 & \dots & 0 \\ N_2C_2 & 0 & \Phi_2 & \ddots & \vdots & \vdots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & \vdots & & \vdots \\ N_\nu C_\nu & 0 & \dots & 0 & \Phi_\nu & 0 & & \vdots \\ L_1C_1 & 0 & \dots & & 0 & \Psi_1 & \ddots & \vdots \\ \vdots & \vdots & & & & \ddots & \ddots & 0 \\ L_\nu C_\nu & 0 & \dots & \dots & \dots & \dots & 0 & \Psi\nu \end{bmatrix}$$

Hurwitz. In other words, the system is rendered exponentially stable in the first approximation. Moreover, by hypothesis there exist mappings $x = \pi(w)$ and $\xi_i = \sigma_i(w), 1 \le i \le \nu$, with $\pi(0) = 0$, $\sigma_i(0) = 0$, $1 \le i \le \nu$, and $\xi_i = \tau_i(w)$ such that (8) hold together with (because of the immersion) $(\partial\tau_i(w)/\partial w)s(w) = \varphi_i(\tau_i(w))$ and $c_i(w) = \gamma_i(\tau_i(w))$, $1 \le i \le \nu$. Then (8) together with $\xi_i = \tau_i(w), \chi_i = 0$, $1 \le i \le \nu$, satisfy (6). This, on the other hand, implies that the sufficient conditions of lemma 1 are satisfied (i.e., we have local controller that satisfies conditions **(S)** and (6) are satisfied), and therefore, output regulation is achieved and this completes the proof of the theorem. □

One issue to note here is that the above result does not specify how the stabilizing controller should be designed. In fact, as long as the interconnection of the system in (1) with the local (partial) internal model in (10) (with the choice of appropriate set of matrices N_i, $i = 1, \dots, \nu$) is locally stabilizable via decentralized control there might be several possible options for developing such a controller and the designer is free to use any method.

From the above formulation, we have that each of our local controllers have the following form

$$\begin{aligned} \dot{\xi}_i &= \varphi_i(\xi_i) + N_i e_i, \\ \dot{\chi}_i &= \Psi_i \chi_i + L_i e_i, \\ u_i &= \gamma_i(\xi_i) + M_i \chi_i, \ 1 \le i \le \nu. \end{aligned} \tag{14}$$

In other words, each of the local controllers consists of a parallel connection of a dynamic compensator and a servocompensator similar to the centralized case.

Since the above result is stated in terms of the fixed modes of the cascade connection of the plant and the servocompensator, one wonders how do we determine or characterize the decentralized fixed modes of a given system. Since the introduction of the concept of fixed modes in Wang and Davison (1973), several authors have addressed this issue. One may consult Anderson and Clements (1981), Davison and Wang (1985), Xu *et al.* (1988), and Gong and Aldeen (1992) and references therein for relevant discussions.

3. Output regulation of a class of interconnected systems

In this section, we consider the output regulation problem of the class of interconnected systems which consist of ν interconnected subsystems. Figure 2 shows an illustrative example of such an interconnected system which has six subsystems. We assume that the dynamics of the subsystems are described by

$$\begin{aligned} \dot{x}_i &= f_i(x_1, \dots, x_\nu, w_i, u_i), \\ e_i &= h_i(x_i, w_i), \end{aligned} \tag{15}$$

for $1 \le i \le \nu$, where $x_i \in \mathbb{R}^{n_i}$ $(n = \sum_{i=1}^{\nu} n_i)$ represent the local state of each subsystem, $u_i \in \mathbb{R}^{m_i}$ and $e_i \in \mathbb{R}^{m_i}$, $1 \le i \le \nu$, $(m = \sum_{i=1}^{\nu} m_i)$ are the local control inputs and outputs, respectively. The signal $w_i \in \mathbb{R}^{r_i}$ $(r = \sum_{i=1}^{\nu} r_i)$ are the local exogenous inputs to each subsystem (as shown in figure 2) and are generated by neutrally stable exosystems

$$\dot{w}_i = s_i(w_i), 1 \le i \le \nu. \tag{16}$$

We assume that all the above functions f_i, h_i, and s_i are known and smooth. As in the preceding section, the objective is to design a decentralized regulator that uses only local controls that will provide asymptotic regulation of the output of each of the subsystems to zero. In other words, the system together with its

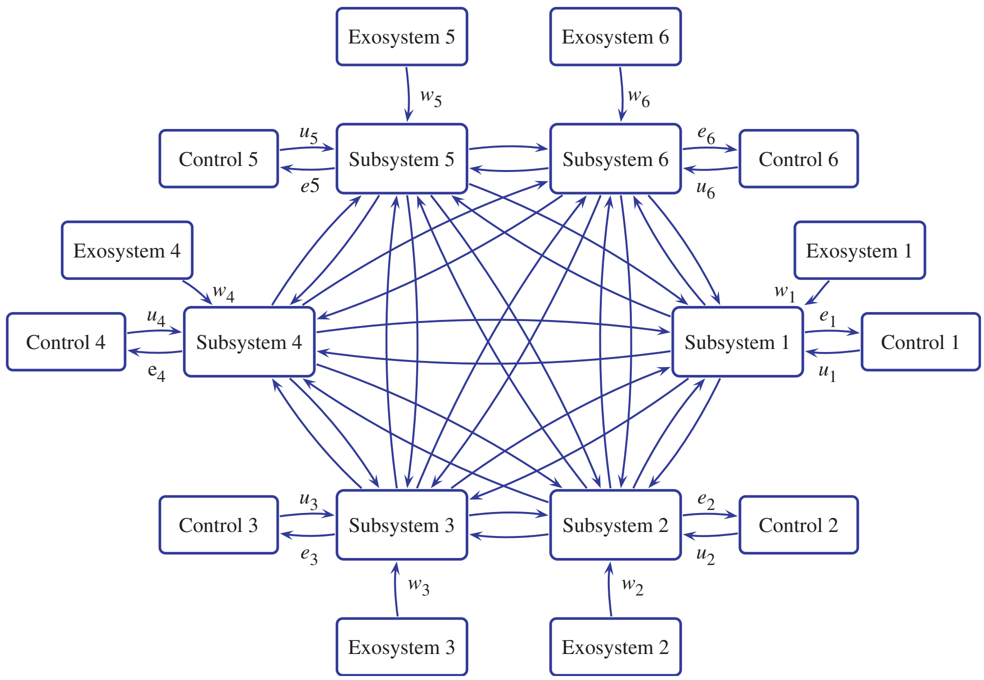

Figure 2. Interconnected system with ν subsystems.

exogeneous inputs and local controls will have a structure similar to to the one shown in figure 2.

From the earlier analysis we know that a necessary condition for the existence of a solution of this problem is the existence of mappings $x_i = \pi_i(w_i)$ and $u_i = c_i(w_i)$, $1 \le i \le \nu$, with $\pi_i(0) = 0$ and $c_i(0) = 0$, $1 \le i \le \nu$, defined in a neighborhood W_i^o of the origin of $\mathbb{R}^{r_i}$, respectively, such that for each $i = 1, \dots, \nu$ we have

$$\begin{aligned} \frac{\partial \pi_i}{\partial w_i} s_i(w_i) &= f_i(\pi_1(w_1), \dots, \pi_\nu(w_\nu), w_i, c_i(w_i)), \\ 0 &= h_i(\pi_i(w_i), w_i), \end{aligned} \tag{17}$$

for all $w_i \in W_i^o$, respectively.

One can easily see that it is possible to establish a counterpart of lemma 1 also for this case. Therefore, we do not present such a result here. Still, let us denote the system matrices of the first approximation of the subsystems with $A_i = (\partial f_i/\partial x_i)(0, \dots, 0, 0, 0)$, $E_{i,j} = (\partial f_i/\partial x_j)(0, \dots, 0, 0, 0)$, $B_i = (\partial f_i/\partial u_i), (0, \dots, 0, 0, 0)$, and $C_i = (\partial h_i/\partial x_i)(0, 0)$, and also define

$$A = \begin{bmatrix} A_1 & E_{1,2} & \dots & E_{1,\nu} \\ E_{2,1} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & E_{\nu-1,\nu} \\ E_{\nu,1} & \dots & E_{\nu,\nu-1} & A_\nu \end{bmatrix}, \quad \begin{aligned} B &= bd[B_1, \dots, B_\nu], \\ C &= bd[C_1, \dots, C_\nu], \end{aligned} \tag{18}$$

where *bd* stands for block diagonal as was mentioned before. Then, temporarily assuming $w = 0$ (i.e., temporarily ignoring the exogeneous inputs) the linearization (or the first approximation) of the system in (15) around the origin can be represented as

$$\begin{aligned} \dot{x}_i &= A_i x_i + \sum_{j=1, j\neq i}^{\nu} E_{i,j} x_j + B_i u_i \\ y_i &= C_i x_i, \end{aligned} \tag{19}$$

for $1 \le i \le \nu$. This approximation of the system is similar to those considered in Davison (1976) and the discussions there are relevant/applicable. In order to be able to design the regulator we need conditions on the stabilizability of the system using decentralized control. Therefore, to establish the main result of this section we use the following lemma that is taken (actually deduced) from Davison (1976) (see theorem 6 and remark 6 – we present it here for the convenience of the reader).

Lemma 2 Consider the interconnected composite system in (19) and assume that each of the subsystems (C_i, A_i, B_i), $i = 1, \dots, \nu$, are all stabilizable and detectable. Then, there exists a scalar $E > 0$, such that for the class of nonzero interconnection gains $E_{i,j}$ satisfying $\|E_{i,j}\| < E$, $i = 1, \dots, \nu$, $j = 1, \dots, \nu$, $i \neq j$, the system is stabilizable via decentralized control.

This lemma makes intuitive sense. Since each of the subsystems is stabilizable and detectable and for

$E_{i,j}=0$ for all $1 \le i \le \nu$, $1 \le j \le \nu$, $i \ne j$, the system is decoupled we can find an appropriate local output feedback such that all the eigenvalues of the closed loop system would have negative real parts. Therefore, if the interconnections are sufficiently weak, then the stability properties of the closed loop system matrix are preserved.

An issue to note here is that in many applications the level of interconnections between the subsystems will be given and one will not be able to choose E. This limits the applicability of lemma 2 to the systems in which the condition $\|E_{i,j}\| < E$, $i=1,\dots,\nu$, $j=1,\dots,\nu$, $i \ne j$ is satisfied *a prior*. Nevertheless, there are applications such as coordination and control or formation control of multi-agent (multi-robot) systems in which it might be possible to choose/design the interconnections/interactions between the subsystems. This is because in multi-agent systems each agent has its own dynamics and the "coupling" in the system arises due to the coordination strategy. In other words, the coordination strategy brings the restriction that the motion of the agents is affected by the motion of the other agents in order to achieve the overall group objective resulting in a system which can be viewed as an interconnected system. The flexibility in such systems is that usually there might be several coordination strategies achieving the same objective and the designer can choose the coordination strategy with weaker "coupling" between the agent motion dynamics.

Let $\bar{E}=\sup\{E\}$, where E is as defined in lemma 2. In other words, $\bar{E}$ is the maximum possible "coupling" that the system can tolerate and still be stabilized via decentralized control. With this definition we are ready to state the main result of this section.

Theorem 2: The DORP for the interconnected system in (16) is solvable if

(1) There exist mappings $x_i=\pi_i(w_i)$ and $u_i=c_i(w_i)$, $1 \le i \le \nu$, with $\pi_i(0)=0$ and $c_i(0)=0$, $1 \le i \le \nu$, defined in a neighborhood W_i^o of the origin of $\mathbb{R}^{r_i}$, respectively, such that (17) are satisfied for all $w_i \in W_i^o$, respectively.

(2) For all $i=1,\dots,\nu$, the autonomous systems

$$\begin{aligned} \dot{w}_i &= s_i(w_i), \\ u_i &= c_i(w_i), \end{aligned} \tag{20}$$

are immersed into

$$\begin{aligned} \dot{\xi}_i &= \varphi(\xi_i), \\ u_i &= \gamma_i(\xi_i), \end{aligned} \tag{21}$$

defined on neighborhoods Ω_i of the origins of $\mathbb{R}^{p_i}$, respectively, in which $\varphi_i(0)=0$ and $\gamma_i(0)=0$.

(3) For all $i=1,\dots,\nu$, the matrices

$$\Phi_i = \left[\frac{\partial \varphi_i}{\partial \xi_i}\right]_{\xi_i=0} \quad \text{and} \quad \Gamma_i = \left[\frac{\partial \gamma_i}{\partial \xi_i}\right]_{\xi_i=0}$$

are such that each of the pairs

$$\begin{bmatrix} A_i & 0 \\ N_i C_i & \Phi_i \end{bmatrix}, \begin{bmatrix} B_i \\ 0 \end{bmatrix}, \tag{22}$$

is stabilizable for some N_i and each of the pairs

$$\begin{bmatrix} C_i & 0 \end{bmatrix}, \begin{bmatrix} A_i & B_i\Gamma_i \\ 0 & \Phi_i \end{bmatrix}, \tag{23}$$

is detectable.

(4) The interconnections satisfy $\|E_{i,j}\| \le \bar{E}$.

Proof We show that the conditions above satisfy the sufficient conditions of lemma 1. To this end, choose the matrices N_i, $1 \le i \le \nu$, such that the pairs in (22) are stabilizable. Now, note that each of the triples

$$\begin{bmatrix} C_i & 0 \end{bmatrix}, \begin{bmatrix} A_i & B_i\Gamma_i \\ N_i C_i & \Phi_i \end{bmatrix}, \begin{bmatrix} B_i \\ 0 \end{bmatrix},$$

is stabilizable and detectable. Therefore, there exist matrices Ψ_i, L_i, and M_i, $i \le i \le \nu$, such that the matrices

$$\bar{A}_i = \begin{bmatrix} \begin{bmatrix} A_i & B_i\Gamma_i \\ N_i C_i & \Phi_i \end{bmatrix} & \begin{bmatrix} B_i \\ 0 \end{bmatrix} M_i \\ L_i \begin{bmatrix} C_i & 0 \end{bmatrix} & \Psi_i \end{bmatrix}$$

have their eigenvalues in the open left-half plane. Define

$$\bar{E}_{i,j} = \begin{bmatrix} E_{i,j} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then, the linearization of the closed loop system equations become

$$\begin{bmatrix} \bar{A}_1 & \bar{E}_{1,2} & \dots & \bar{E}_{1,\nu} \\ \bar{E}_{2,1} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \bar{E}_{\nu-1,\nu} \\ \bar{E}_{\nu,1} & \dots & \bar{E}_{\nu,\nu-1} & \bar{A}_\nu \end{bmatrix}.$$

Since the interconnections satisfy $\|E_{i,j}\| \leq \bar{E}$, from lemma 2 we know that there exists a decentralized controller that stabilizes the system. In other words, above we can choose the matrices L_i, M_i, and Ψ_i such that the closed loop system is stable. This proves that the condition **(S)** is satisfied. This together with the other hypotheses of the theorem satisfy the sufficiency conditions of lemma 1 (or an equivalent modified version of lemma 1 for interconnected systems), and this completes the proof. □

4. Example: car following in automated highway system

In this section we provide an application example in order to illustrate the procedure. In particular, we consider the problem of car following in an Automated Highway System (AHS) (Spooner and Passino 1996). In AHS the vehicles will be automatically driven by onboard computers. One possible method to implement the controllers responsible for automatically driving the vehicles can be to use lateral and longitudinal controllers in parallel, where lateral controllers will be responsible for tasks such as steering the vehicle and changing lanes, whereas the longitudinal controllers will be used for tasks such as maintaining steady velocity and following a vehicle in front (on the same lane) at a safe distance. Here we consider the car following problem (possibly in a platoon of vehicles). Since platooning (having the vehicles move close to each other at constant inter-vehicle distances and with constant velocities) may lead to fuel savings, safer driving, and higher traffic throughput, it may be desirable in AHS. The dynamics of the i'th vehicle in a car following system can be described by Spooner and Passino (1996)

$$\begin{aligned} \dot{\psi}_i &= v_i - v_{i-1}, \\ \dot{v}_i &= \frac{1}{m_i}\Big[-A^i_\rho v_i^2 - d_i + f_i\Big], \\ \dot{f}_i &= \frac{1}{\tau_i}[-f_i + u_i], \end{aligned} \tag{26}$$

where $\psi_i = X_i - X_{i-1}$ is the inter-vehicle spacing between the i'th and the $(i-1)$'th vehicles (X_i shows the longitudinal position of the i'th vehicle), v_i is the longitudinal velocity (speed) of the i'th vehicle, and f_i is the driving/braking force applied to the longitudinal dynamics of the i'th vehicle. The variable u_i is the throttle/brake input applied to the vehicle where $u_i > 0$ represents a throttle input, whereas $u_i < 0$ represents a brake input. The parameter m_i represents the mass, A^i_ρ represents the aerodynamic drag constant, d_i represents the constant frictional force, and τ_i represents the engine/brake time constant of the i'th vehicle, respectively. The output (error) of the system is chosen as

$$e_i = \psi_i + \rho_i v_i + L_i,$$

where L_i and ρ_i are positive constants. This output allows for velocity dependent spacing due to the $\rho_i v_i$ term in addition to the constant distance L_i. Setting $\rho_i = 0.9$ allows 9 m extra spacing for every 10 m/s extra speed. We assume that the inter-vehicle distances ψ_i are measurable and the objective is to regulate e_i to zero. As one can easily see this is an interconnected system and the results developed in the preceding sections can be applied. Defining the state of the system as $x_i = [\psi_i, v_i, f_i]^\top$ and the exogeneous input as $w_i = [d_i, L_i]^\top$ and assuming that the constant reference velocity for the leading vehicle is v_0 (which can be set/achieved by a classic cruise control system) one can easily see that the mappings $\pi_i(w_i)$ and $c_i(w_i)$ (for the follower vehicles) are given by

$$\pi_i(w_i) = \begin{bmatrix} -\rho_i v_0 - L_i \\ v_0 \\ A^i_\rho v_0^2 + d_i \end{bmatrix} \text{ and } c_i(w_i) = A^i_\rho v_0^2 + d_i.$$

The above zero-error manifold $\pi_i(w_i)$ and the corresponding controller (the so-called "friend") $c_i(w_i)$ are achieved when all the vehicles are moving with constant velocities equal to the velocity of the leading vehicle $v_i = v_0$ and at constant inter-vehicle spacings of $\psi_i = -\rho_i v_0 - L_i$.

It might be possible to design the regulating controller using nonlinear techniques. However, in order to be consistent with the analysis in the preceding sections we need to linearize the model. Linearizing the dynamics around an operating point at $v_i = v_0 = constant$ (for example around the point $\bar{x}_i = [0, v_0, 0]^\top$ – note that this does not change the results since we can always redefine the state so that this point corresponds to the origin) we obtain

$$\begin{aligned}\dot{x}_i &= A_i x_i + E_{i,i-1} x_{i-1} + F_i w_i + B_i u_i,\\ e_i &= C_i x_i + D_i w_i, \\ \dot{w}_i &= 0\end{aligned} \tag{25}$$

where

$$A_i = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -\frac{2A_\rho^i v_0}{m_i} & \frac{1}{m_i} \\ 0 & 0 & -\frac{1}{\tau_i}\end{bmatrix}, \quad E_{i,i-1} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{bmatrix},$$

$$F_i = \begin{bmatrix} 0 & 0 \\ -\frac{1}{m_i} & 0 \\ 0 & 0 \end{bmatrix}, B_i = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{\tau_i}\end{bmatrix},$$

$C_i = \begin{bmatrix} 1 & \rho_i & 0 \end{bmatrix}$, and $D_i = \begin{bmatrix} 0 & 1 \end{bmatrix}$.

As mentioned above here we assume that the constant "inputs" L_i are known and the inter-vehicle distances ψ_i are measurable. In other words, we assume that the output errors e_i are measurable. In contrast, we do not know the constant disturbances d_i and the velocity of the preceding vehicle. Since $c_i(w_i)$ depends only on d_i (which is the only unknown exogeneous input) we can choose the immersion as one dimensional. In other words, we can choose $\Phi_i = [0]$ (since d_i's are constants), $\Gamma_i = [1]$, and $N_i = [1]$ for all i.

Using these the system matrices of the interconnection of the local internal model with the vehicle dynamics become

$$\tilde{A}_i = \begin{bmatrix} A_i & B_i\Gamma_i \\ N_i C_i & \Phi_i \end{bmatrix}, \tilde{B}_i = \begin{bmatrix} B_i \\ 0 \end{bmatrix}, \text{ and } \tilde{C}_i = \begin{bmatrix} C_i & 0 \end{bmatrix},$$

which is a four-dimensional system. Given these system matrices the next step is to design a stabilizing controller (the servocompensator). As was mentioned before one may design the stabilizing controller using different methods. One possible method is to use an observer, based state feedback controller (Khalil 1996) in which Ψ_i can be calculated as

$$\Psi_i = \tilde{A}_i - \tilde{B}_i M_i - L_i \tilde{C}_i$$

where the L_i and the M_i are the gain vectors/matrices of the state feedback and the observer can be determined by placing the eigenvalues of $(\tilde{A}_i - \tilde{B}_i M_i)$ and $(\tilde{A}_i - L_i\tilde{C}_i)$ at the desired locations. In particular, one can calculate them using the Matlab command `place`

$$\begin{aligned} L_i^\top &= place(\tilde{A}_i, \tilde{C}_i^\top, P_i^1) \\ M_i &= place(\tilde{A}_i, \tilde{B}_i, P_i^2)\end{aligned}$$

for the sets of desired poles specified in the vectors P_i^1 and P_i^2. In the simulations below we used this type of observer-based stabilizing controller for which we specified $P_i^1 = [-1, -2, -3, -4]$ and $P_i^2 = [-0.5, -1.0, -1.5, -2.0]$ as the desired pole locations for the state feedback and the observer parts which resulted in the matrices

$$L_i = \begin{bmatrix} -0.7867 \\ 6.4297 \\ -0.5880 \\ 5.9160 \end{bmatrix}, \quad M_i = \begin{bmatrix} 1274 & 2275 & 0 & 391 \end{bmatrix},$$

$$\text{and } \Psi_i = \begin{bmatrix} 1 & 2 & 0 & 0 \\ -6 & -6 & 0 & 0 \\ -6369 & -11374 & -5 & -195 \\ -5 & -4 & 0 & 0 \end{bmatrix}.$$

These matrices are achieved independent of the value of the constant velocity v_0 at the operating point around which the linearization A_i is obtained. In other words, the above controller matrices and the closed loop poles are the same for $v_0 = 0\,\text{m/s}$ and $v_0 = 25\,\text{m/s}$.

As vehicle parameters we used the values $m_i = 1300\,\text{kg}$, $A_\rho^i = 0.3\,\text{Ns}^2/\text{m}^2$, $\tau_i = 0.2\,\text{s}$ for all the vehicles. Moreover we used the values $\rho_i = 0.9$ as required inter-vehicle distance parameter, $d_i = 100\,\text{N}$ as the unknown constant friction disturbance and $L_i = 10$ as reference input.

We simulated the behavior of the system in a platoon of six vehicles. We assumed that the leader in the platoon is moving with a constant velocity of 25 m/s and initiated the other five vehicles with zero initial velocity and at a position 100 m behind its predecessor.

Figure 3 shows the plot of the inter-vehicle distances in the platoon. As is easily seen from the figure the inter-vehicle distances settle very quickly to a constant value. In fact, they asymptotically converge to the value of 32.5 m, which is the desired inter-vehicle spacing achieved at zero output errors. Figure 4 shows the plot of the output errors $e_i(t)$. As seen from the figure the control objective to regulate these outputs to zero is asymptotically achieved supporting the discussions/results in the preceding sections.

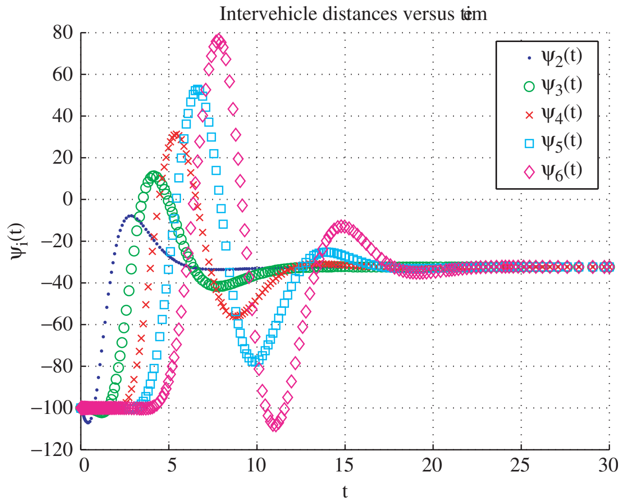

Figure 3. Inter-vehicle distances $\psi_i(t)$ versus time t.

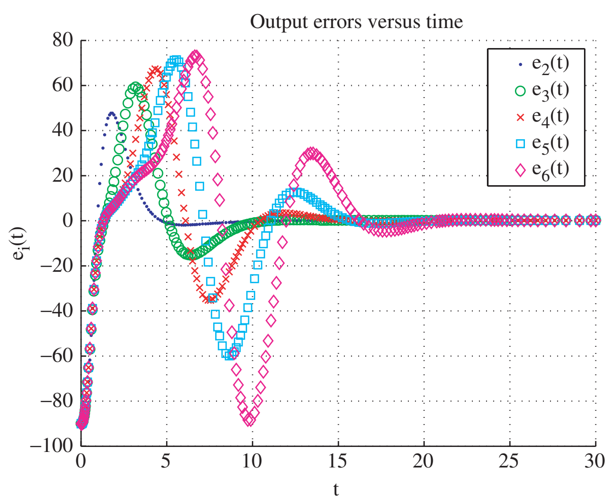

Figure 4. Output errors $e_i(t)$ versus time t.

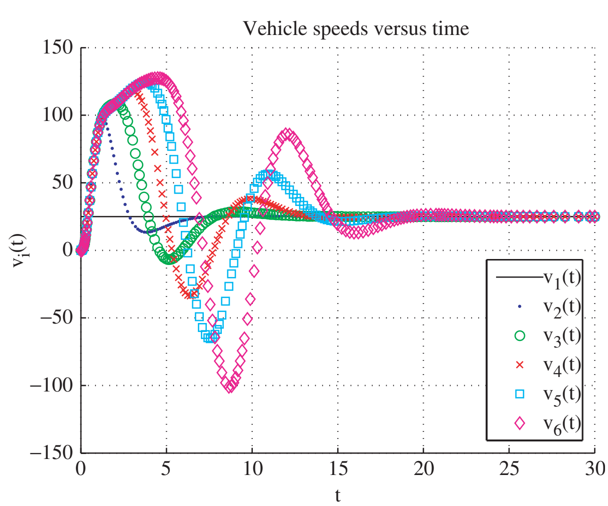

Figure 5. Vehicle speeds $v_i(t)$ versus time t.

From the discussion in the preceding sections we expect that the state of the system will converge to the zero error manifold $\pi_i(w_i)$ where the velocities of the vehicles will stabilize at the reference velocity (set by the leading vehicle) of 25 m/s. From the plot of the longitudinal speeds of the vehicles, shown in figure 5, one can easily see that this expectation (or objective) is achieved.

One disadvantage of using an observer-based stabilizing controller may be that initially there may be large discrepancy between the states of the observer and those of the plant (and this is in addition to the discrepancy between the states of the immersion and the exosystem). Therefore, initially we may have large control inputs. However, it is not required to use an observer-based state feedback as a stabilizing controller and one may use controllers designed using other techniques as well.

5. Conclusions

The problem of output regulation in presence of uncertainties and disturbances is an important problem in control theory. It has been extensively studied and solved for linear systems, and for certain classes of nonlinear systems. Efforts to develop conditions for the solvability of the problem in a semi-global or global sense for more general classes of nonlinear systems, as well as developing effective controllers for these, still continue. In this article we presented conditions for the local solution of the problem for a class of decentralized and interconnected systems using decentralized controllers. The results presented can easily be extended to structurally stable regulation, output regulation of discrete-time systems, and output regulation using adaptive internal model.

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