Stable Social Foraging Swarms in a Noisy Environment

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***Abstract*—Bacteria, bees, and birds often work together in groups to find food. A group of robots can be designed to coordinate their activities to search for and collect objects. Networked cooperative uninhabited autonomous vehicles are being developed for commercial and military applications. Suppose that we refer to all such groups of entities as "social foraging swarms." In order for such multiagent systems to succeed it is often critical that they can both maintain cohesive behaviors and appropriately respond to environmental stimuli (e.g., by optimizing the acquisition of nutrients in foraging for food). In this paper, we characterize swarm cohesiveness as a stability property and use a Lyapunov approach to develop conditions under which local agent actions will lead to cohesive foraging even in the presence of "noise" characterized by uncertainty on sensing other agent's position and velocity, and in sensing nutrients that each agent is foraging for. The results quantify earlier claims that social foraging is in a certain sense superior to individual foraging when noise is present, and provide clear connections between local agent-agent interactions and emergent group behavior. Moreover, the simulations show that very complicated but orderly group behaviors, reminiscent of those seen in biology, emerge in the presence of noise.**

***Index Terms*—Biological systems, foraging, multiagent systems, stability analysis, swarming.**

I. Introduction

SWARMING has been studied extensively in biology [1], [2], and there is significant relevant literature in physics where collective behavior of "self-propelled particles" is studied. Swarms have also been studied in the context of engineering applications, particularly in collective robotics where there are teams of robots working together by communicating over a communication network [3], [4]. For example, the work in [5] on "social potential functions" is similar to how we view attraction-repulsion forces. Special types of swarms have been studied in "intelligent vehicle highway systems" [6] and in "formation control" for robots, aircraft, and cooperative control for uninhabited autonomous (air) vehicles. Early work on swarm stability is in [7], [8]. Later work where there is asynchronism and time delays appears in [9]–[11]. Also relevant is the study in [12] where the authors use virtual leaders and artificial potentials.

In this paper, we continue some of our earlier work by studying stability properties of foraging swarms. The main difference with our previous work is that we consider the effect of sensor errors ("noise") and errors in sensing the gradient of a "resource profile" (e.g., a nutrient profile). We are able to show that even with noisy measurements the swarm can achieve cohesion and follow a nutrient profile in the proper direction. We illustrate that the agents can forage in noisy environments more efficiently as a group than individually, a principle that has been identified for some organizms [13], [14]. The work here builds on the work in [15] and [16] where the authors provide a class of attraction/repulsion functions and provide conditions for swarm stability (ultimate swarm size and ultimate behavior), and [17] that represents progress in the direction of combining the study of aggregating swarms and how during this process decisions about foraging or threat avoidance can affect the collective/individual motion of the swarm/swarm members (i.e., typical characteristics influencing social foraging). Additional work on gradient climbing by swarms, including work on climbing noisy gradients, is in [18]. There, similar to [17], the authors study climbing gradients, but also consider noise effects and coordination strategies for climbing, something that we do not consider here.

The remainder of this paper is organized as follows. In Section II, we introduce a generic model for agents, interactions, and the foraging environment. Section III holds the main results on stability analysis of swarm cohesion. Stability analysis of swarm trajectory following is done in Section IV. Section V holds the simulation results and some concluding remarks are provided in Section VI.

II. Swarm and Environment Models

A. Agents and Interactions

Here, rather than focusing on the particular characteristics of one type of animal or autonomous vehicle we consider a swarm composed of an interconnection of N "agents," each of which has point mass dynamics given by

$$\dot{x}^i = v^i$$
$$\dot{v}^i = \frac{1}{M_i} u^i \tag{1}$$

where $x^i \in \Re^n$ is the position, $v^i \in \Re^n$ is the velocity, M_i is the mass, and $u^i \in \Re^n$ is the (force) control input for the ith agent. It is assumed that all agents know their own dynamics. For some organizms like bacteria that move in highly viscous environments you can assume that $M_i = 0$ and if you use a velocity damping term in u^i for this you get the model studied in [15]–[17]. There, the authors view the choice of u^i as one that seeks to perform "energy minimization" which is consistent with other energy formalisms in mathematical biology. Here,

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we do not assume $M_i = 0$. Moreover, we will assume that each agent can sense information about the position and velocity of other agents, but possibly with some errors (what we will call "noise"), something not considered in [15]–[17].

Agent to agent interactions considered here are of the "attract-repel" type where each agent seeks to be in a position that is "comfortable" relative to its neighbors (and for us all other agents are its neighbors). Attraction indicates that each agent wants to be close to every other agent and it provides the mechanism for achieving grouping and cohesion of the group of agents. Repulsion provides the mechanism where each agent does not want to be too close to any other agent (e.g., for animals to avoid collisions and excessive competition for resources). There are many ways to define attraction and repulsion, each of which can be represented by characteristics of how we define u^i for each agent. Attraction here will be represented by a term in u^i like $-k_p^i\left(x^i - x^j\right)$ where $k_p^i > 0$ is a scalar that represents the strength of attraction. If the agents are far apart, then there is a large attraction between them, and if they are close there is a small attraction. For repulsion, we let the two-norm $\|z\| = \sqrt{z^\top z}$ and use a repulsion term in u^i of the form

$$k_r^i \exp\left(\frac{-\frac{1}{2}\|x^i - x^j\|^2}{{r_s^i}^2}\right)\left(x^i - x^j\right) \tag{2}$$

where $k_r^i > 0$ is the magnitude of the repulsion, and $r_s^i > 0$ quantifies the region size around the agent from which it will repel its neighbors. When $\|x^i - x^j\|$ is big relative to r_s^i the whole term approaches zero. The combined effect of repulsion and attraction terms influence the so-called "equilibrium of attraction and repulsion" that represents that two agents are at a "comfortable" distance from one another. Many other types of attraction and repulsion terms are possible; see the references cited earlier.

B. Environment Model

Next, we will define the environment that the agents move in. While there are many possibilities, here we will simply consider the case where they move (forage) over a "resource profile" (e.g., nutrient profile) $J(x)$, where $x \in \Re^n$. We will assume that $J(x)$ is continuous with finite slope at all points. Agents move in the direction of the negative gradient of $J(x)$ (i.e., in the direction of $-\nabla J(x) = -(\partial J/\partial x)$) in order to move away from "bad" areas and into "good" areas of the environment (e.g., to avoid noxious substances and find nutrients). That is, they will use a term in their u^i that holds the negative gradient of $J(x)$.

Clearly there are many possible shapes for $J(x)$, including ones with many peaks and valleys. Here, we simply list two simple forms for $J(x)$ as follows.

- *Plane:* In this case, we have $J(x) = J_p(x)$ where

$$J_p(x) = R^\top x + r_p$$

  where $R \in \Re^n$ and r_p is a scalar. Here, $\nabla J_p(x) = R$.
- *Gaussian:* In this case, we have $J(x) = J_g(x)$ where

$$J_g(x) = r_{m_1} \exp\left(-r_{m_2}\|x - R_c\|^2\right) + r_e$$

  where r_{m_1}, r_{m_2} and r_e are scalars, $r_{m_2} > 0$ and $R_c \in \Re^n$. Here, $\nabla J_e(x) = -2r_{m_1}r_{m_2}\exp\left(-r_{m_2}\|x - R_c\|^2\right)(x - R_c)$.

Here, we will assume that each agent can sense the gradient, but only with some errors, which we will refer to as "noise." When we want to refer to agent i as following a possibly different profile than agent j, $j \neq i$, we will use, for example, J_p^i, R^i, and r_p^i for the plane profiles. Throughout this paper, we will in fact study this case where agents follow different plane profiles, since these are simple yet representative. Moreover, locally, typical profiles will have a constant slope.

III. Stability Analysis of Swarm Cohesion Properties

Cohesion and swarm dynamics will be quantified and analyzed using stability analysis of the swarm error dynamics that we define next.

A. Controls and Error Dynamics

First, assume there are N agents in the environment and let

$$\bar{x} = \frac{1}{N}\sum_{i=1}^{N} x^i$$
$$\bar{v} = \frac{1}{N}\sum_{i=1}^{N} v^i$$

be the average position and velocity of the swarm, respectively. The objective of each agent is to move so as to end up at $\bar{x}$ and $\bar{v}$; in this way an emergent behavior of the group is produced where they aggregate dynamically and end up near each other and ultimately move in the same direction at nearly the same velocity (i.e., cohesion). The problem is that since all the agents are moving at the same time, $\bar{x}$ and $\bar{v}$ are time-varying; hence, in order to study the stability of swarm cohesion we study the dynamics of an error system with

$$e_p^i = x^i - \bar{x}$$
$$e_v^i = v^i - \bar{v}.$$

Other choices for error systems are also possible.

The error dynamics are given by

$$\dot{e}_p^i = e_v^i$$
$$\dot{e}_v^i = \frac{1}{M_i}u^i - \frac{1}{N}\sum_{j=1}^{N}\frac{1}{M_j}u^j. \tag{3}$$

We assume that each agent can sense its own position and velocity relative to $\bar{x}$ and $\bar{v}$, but with some errors. In particular, let $d_p^i \in \Re^n$ and $d_v^i \in \Re^n$ be these sensing errors for agent i, respectively. We assume that $d_p^i(t)$ and $d_v^i(t)$ are any trajectories that are sufficiently smooth and fixed a priori for all the time (but below we will study the stability for the case when the d_p^i and d_v^i trajectories can be any of a certain class). We will refer to these terms somewhat colloquially as "noise" but clearly our framework is entirely deterministic. Thus, each agent actually senses

$$\hat{e}_p^i = e_p^i - d_p^i$$
$$\hat{e}_v^i = e_v^i - d_v^i$$

and below we will also assume that it can sense its own velocity. It is important to highlight here our motivation for studying the addition of noise. On the one hand it adds another element of realism to have a sensor for e_p^i and e_v^i might operate and the results below will help quantify the effects of the noise on cohesion. We also view our approach as progress in the direction of not requiring that each agent can sense the variables of all the other agents or even the accurate values of $\bar{x}$ and $\bar{v}$.

The ith agent will also try to follow a plane nutrient profile J_p^i defined earlier. We assume that it senses the gradient of J_p^i, but with some sufficiently smooth error $d_f^i(t)$ that is fixed *a priori* for all the time (as with d_p^i and d_v^i we will allow below d_f^i to be any in a certain class of trajectories) so each agent actually senses

$$\nabla J_p^i\left(x^i\right) - d_f^i.$$

In fact this can be viewed as either sensing error or "noise" (variations, ripples) on the resource profile.

Suppose the general form of the control input for each agent is

$$\begin{aligned} u^i =& -M_i k_p^i \hat{e}_p^i - M_i k_v^i \hat{e}_v^i - M_i k v^i \\ &+ M_i k_r^i \sum_{j=1, j\neq i}^{N} \exp\left(\frac{-\frac{1}{2}\left\|\hat{e}_p^i - \hat{e}_p^j\right\|^2}{{r_s^i}^2}\right)\left(\hat{e}_p^i - \hat{e}_p^j\right) \\ &- M_i k_f^i \left(R^i - d_f^i\right). \end{aligned} \tag{4}$$

Here, we think of the scalars $k_p^i > 0$ and $k_v^i > 0$ as the "attraction gains" which indicate how aggressive each agent is in aggregating. The gain $k_r^i > 0$ is a "repulsion gain" which sets how much that agent wants to be away from others and r_s^i represents its repulsion range. The gain $k > 0$ works as a "velocity damping gain"(note that we use the same such gain for all agents). The last term in (4) indicates that each agent wants to move along the negative gradient of the ith resource profile with the gain $k_f^i > 0$ proportional to that agent's desire to follow its profile. Obviously if $d_p^i = 0$ for all i, there is no sensing error on repulsion, and a repulsion term of the form explained in (2) is obtained. The sensing errors create the possibility that agents will try to move away from each other when they may not really need to, and they may move toward each other when they should not. Similarly, the attractions gains k_p^i and k_v^i dictate how the attraction forces operate but the presence of the noise results in additive noise terms to u^i that are multiplied by k_p^i and k_v^i. Hence, raising the attraction gains also has a negative influence on u^i in that it results in more noise entering the control and hence poor aggregating decisions by individuals (e.g., if $\|x^i - x^j\|$ is small but d_p^i is relatively large, noise will set the control value). Clearly, this complicates the situation for the whole swarm to achieve cohesiveness.

Note that by writing the repulsion term as in (4), we are assuming each agent can also sense the positions of all other agents relative to $\bar{x}$; however, the sensed values for other agents are only needed in the repulsion term and any term corresponding to a distant agent will be close to zero due to the exponential term. Alternatively, we may construct this term by replacing $\hat{e}_p^i - \hat{e}_p^j$ with $\hat{x}^i - \hat{x}^j$, with $\hat{x}^i$ and $\hat{x}^j$ defined as the noise-contaminated positions of agent i and j, respectively. Then, we have $\hat{x}^i - \hat{x}^j = x^i - x^j + d_p^{ij}$, d_p^{ij} being the measurement noise. In physical sense, these two options of constructing the repulsion term are significantly different from each other since different variables are required to be measured. But note that

$$\begin{aligned} \hat{e}_p^i - \hat{e}_p^j &= \left(\left(x^i - \bar{x}\right) - d_p^i\right) - \left(\left(x^j - \bar{x}\right) - d_p^j\right) \\ &= \left(\hat{x}^i - \hat{x}^j\right) - \left[d_p^{ij} + \left(d_p^i - d_p^j\right)\right]. \end{aligned}$$

It turns out that in our proof, we will obtain the same stability properties with either option. A quick explanation is that the repulsion term is bounded by the same constants (in both directions), whether we adopt $\hat{e}_p^i - \hat{e}_p^j$ or $\hat{x}^i - \hat{x}^j$. This will become more clear by inspecting the proof in Section III-B. From now on, we will use the one in (4) throughout the paper.

To study stability properties, we will substitute the above choice for u^i into the error dynamics in (3). First, consider the $\dot{v}^i$ term of $\dot{e}_v^i = \dot{v}^i - \dot{\bar{v}}$ and note that

$$\begin{aligned} \dot{v}^i =& \frac{1}{M_i} u^i = -k_p^i e_p^i + k_p^i d_p^i - k_v^i e_v^i + k_v^i d_v^i - k v^i \\ &+ k_r^i \sum_{j=1, j\neq i}^{N} \exp\left(\frac{-\frac{1}{2}\left\|\hat{e}_p^i - \hat{e}_p^j\right\|^2}{{r_s^i}^2}\right) \\ &\times \left(\hat{e}_p^i - \hat{e}_p^j\right) - k_f^i \left(R^i - d_f^i\right). \end{aligned} \tag{5}$$

Define $\bar{k}_p = (1/N)\sum_{j=1}^N k_p^j$ and $\Delta k_p^j = k_p^j - \bar{k}_p$. Since

$$\sum_{j=1}^N e_p^j = \sum_{j=1}^N \left(x^j - \bar{x}\right) = N\bar{x} - \sum_{j=1}^N \bar{x} = 0$$

we have

$$\sum_{j=1}^N k_p^j e_p^j = \sum_{j=1}^N \Delta k_p^j e_p^j + \sum_{j=1}^N \bar{k}_p e_p^j = \sum_{j=1}^N \Delta k_p^j e_p^j.$$

Similarly, define $\bar{k}_v$ and Δk_v^j, so we have $\sum_{j=1}^N k_v^j e_v^j = \sum_{j=1}^N \Delta k_v^j e_v^j$. Then, substituting u^i into $\dot{\bar{v}}$ and we have

$$\begin{aligned} \dot{\bar{v}} =& -\frac{1}{N}\sum_{j=1}^N \Delta k_p^j e_p^j + \frac{1}{N}\sum_{j=1}^N k_p^j d_p^j - \frac{1}{N}\sum_{j=1}^N \Delta k_v^j e_v^j \\ &+ \frac{1}{N}\sum_{j=1}^N k_v^j d_v^j - \frac{1}{N}\sum_{j=1}^N k v^j \\ &+ \frac{1}{N}\sum_{l=1}^N k_r^l \sum_{j=1, j\neq l}^N \exp\left(\frac{-\frac{1}{2}\left\|\hat{e}_p^l - \hat{e}_p^j\right\|^2}{{r_s^l}^2}\right)\left(\hat{e}_p^l - \hat{e}_p^j\right) \\ &- \frac{1}{N}\sum_{j=1}^N k_f^j \left(R^j - d_f^j\right). \end{aligned} \tag{6}$$

Define $E^i = \left[{e_p^i}^\top, {e_v^i}^\top\right]^\top$ and $E = \left[{E^1}^\top, {E^2}^\top, \ldots, {E^N}^\top\right]^\top$. Since

$$k v^i - \frac{1}{N}\sum_{j=1}^N k v^j = k v^i - k\bar{v} = k e_v^i$$

from (5) and (6) we have

$$\dot{e}_v^i = \dot{v}^i - \dot{\bar{v}} = -k_p^i e_p^i - \left(k_v^i + k\right) e_v^i + g^i + \phi(E) + \delta^i(E) \tag{7}$$

where

$$g^i = k_p^i d_p^i + k_v^i d_v^i + k_f^i d_f^i - k_f^i R^i \quad (8)$$

$$\phi(E) = \frac{1}{N}\sum_{j=1}^{N} \Delta k_p^j e_p^j + \frac{1}{N}\sum_{j=1}^{N} \Delta k_v^j e_v^j - \frac{1}{N}\sum_{j=1}^{N} k_p^j d_p^j - \frac{1}{N}\sum_{j=1}^{N} k_v^j d_v^j + \frac{1}{N}\sum_{j=1}^{N} k_f^j \left(R^j - d_f^j\right) \quad (9)$$

$$\delta^i(E) = k_r^i \sum_{j=1, j\neq i}^{N} \exp\left(\frac{-\frac{1}{2}\left\|\hat{e}_p^i - \hat{e}_p^j\right\|^2}{{r_s^i}^2}\right)\left(\hat{e}_p^i - \hat{e}_p^j\right) - \frac{1}{N}\sum_{l=1}^{N} k_r^l \sum_{j=1, j\neq l}^{N} \exp\left(\frac{-\frac{1}{2}\left\|\hat{e}_p^l - \hat{e}_p^j\right\|^2}{{r_s^l}^2}\right) \times \left(\hat{e}_p^l - \hat{e}_p^j\right) \quad (10)$$

which is a nonlinear nonautonomous system. With I an $n \times n$ identity matrix, the error dynamics of the ith agent may be written as

$$\dot{E}^i = \overbrace{\begin{bmatrix} 0 & I \\ -k_p^i I & -\left(k_v^i + k\right) I \end{bmatrix}}^{A_i} E^i + \overbrace{\begin{bmatrix} 0 \\ I \end{bmatrix}}^{B} \left(g^i + \phi(E) + \delta^i(E)\right). \quad (11)$$

Note that any matrix

$$\begin{bmatrix} 0 & I \\ -k_1 I & -k_2 I \end{bmatrix}$$

with $k_1 > 0,\ k_2 > 0$ has eigenvalues given by the roots of $\left(s^2 + k_2 s + k_1\right)^n$, which are in the strict left half plane. Thus, the matrix A_i above is Hurwitz with $k_p^i > 0$, $k_v^i > 0$ and $k > 0$.

B. Cohesive Social Foraging With Noise

Our analysis methodology involves viewing the error system in (11) as generating $E^i(t)$ trajectories for a given $E^i(0)$ and the fixed sensing error trajectories $d_p^i(t)$, $d_v^i(t)$, and $d_f^i(t)$, $t \geq 0$. We do not consider, however, all possible sensing error trajectories. We only consider a class of ones that satisfy for all $t \geq 0$

$$\begin{aligned} \|d_f^i(t)\| &\leq D_f^i \\ \|d_p^i(t)\| &\leq D_{p_1}^i \left\|E^i(t)\right\| + D_{p_2}^i \\ \|d_v^i(t)\| &\leq D_{v_1}^i \left\|E^i(t)\right\| + D_{v_2}^i \end{aligned} \quad (12)$$

where $D_{p_1}^i, D_{p_2}^i, D_{v_1}^i, D_{v_2}^i$ and D_f^i are known nonnegative constants for $i = 1, \ldots, N$. So we assume for position and velocity the sensing errors have linear relationship with the magnitude of the state of the error system. Basically the assumption means that when two agents are far away from each other, the sensing errors can increase. The noise d_f^i on the nutrient profile is unaffected by the position of an agent. By considering only this class of fixed sensing error trajectories we prune the set of possibilities for E^i trajectories and it is only for that pruned set that our analysis holds.

1) Uniform Ultimate Boundedness of Interagent Trajectories:

Theorem 1: Consider the swarm described by the model in (3) with control input u^i given in (4). Assume that the nutrient profile for each agent is a plane defined by $\nabla J_p^i(x) = R^i$ and the noise satisfies (12). Let

$$\beta_1^i = \frac{\left(k_p^i + 1\right)^2 + \left(k_v^i + k\right)^2}{2k_p^i\left(k_v^i + k\right)} + \sqrt{\left(\frac{{k_p^i}^2 + \left(k_v^i + k\right)^2 - 1}{2k_p^i\left(k_v^i + k\right)}\right)^2 + \frac{1}{{k_p^i}^2}} \quad (13)$$

for $i = 1, \ldots, N$. If for all i we have

$$k_p^i D_{p_1}^i + k_v^i D_{v_1}^i < \frac{1}{\beta_1^i} \quad (14)$$

and the parameters are such that

$$\sum_{i=1}^{N} \frac{\beta_1^{i^*}\left(k_p^i D_{p_1}^i + k_v^i D_{v_1}^i + \sqrt{{\Delta k_p^i}^2 + {\Delta k_v^i}^2}\right)}{N\left(1 - \theta^i\right)\left(1 - \beta_1^i\left(k_p^i D_{p_1}^i + k_v^i D_{v_1}^i\right)\right)} < 1 \quad (15)$$

for some constants $0 < \theta^i < 1$, where

$$i^* = \arg\max_i \beta_1^i, \qquad i = 1, \ldots, N$$

then the trajectories of (11) are uniformly ultimately bounded.

Proof: To study the stability of the error dynamics, it is convenient to choose Lyapunov function for each agent

$$V_i\left(E^i\right) = {E^i}^\top P_i E^i \quad (16)$$

with $P_i = P_i^\top$ a $2n \times 2n$ matrix and $P_i > 0$ (a positive–definite matrix). Then, we have

$$\begin{aligned} \dot{V}_i &= {E^i}^\top P_i \dot{E}^i + {\dot{E}^i}^\top P_i E^i \\ &= {E^i}^\top \overbrace{\left(P_i A_i + A_i^\top P_i\right)}^{-Q_i} E^i \\ &\quad + 2{E^i}^\top P_i B\left(g^i + \phi^i(E) + \delta^i(E)\right). \end{aligned} \quad (17)$$

Note that when $Q_i = Q_i^\top$ and $Q_i > 0$, the unique solution P_i of $P_i A_i + A_i^\top P_i = -Q_i$ has $P_i = P_i^\top$ and $P_i > 0$ as needed.

Choose for the composite system

$$V(E) = \sum_{i=1}^{N} V_i(E^i)$$

where $V_i\left(E^i\right)$ is given in (16). Since for any matrix $M = M^\top > 0$ and vector X

$$\lambda_{\min}(M) X^\top X \leq X^\top M X \leq \lambda_{\max}(M) X^\top X$$

where $\lambda_{\min}(M)$ and $\lambda_{\max}(M)$ denote the minimum and maximum eigenvalue of M, respectively, from (16) we have

$$\sum_{i=1}^{N}\left(\lambda_{\min}(P_i)\left\|E^i\right\|^2\right) \leq V(E) \leq \sum_{i=1}^{N}\left(\lambda_{\max}(P_i)\left\|E^i\right\|^2\right).$$

It is easy to show that the function $F(\psi) = \exp\left(-1/2\|\psi\|^2/{r_s^i}^2\right)\|\psi\|$, with ψ any real vector, has a unique maximum value of $\exp(-1/2)r_s^i$ which is achieved when $\|\psi\| = r_s^i$ [15]. Defining

$$\Delta^i = k_r^i \exp\left(-\frac{1}{2}\right) \sum_{j=1, j\neq i}^{N} r_s^j + \frac{1}{N}\exp\left(-\frac{1}{2}\right)\sum_{l=1}^{N} k_r^l \sum_{j=1, j\neq l}^{N} r_s^j \tag{18}$$

we have $\left\|\delta^i(E)\right\| \leq \Delta^i$ for $i = 1, \ldots, N$. Define

$$\tilde{R} = \frac{1}{N}\sum_{i=1}^{N} k_f^i R^i.$$

Then, substituting (11) into (17), and using (18) and the fact that $\|B\| = 1$ we have

$$\begin{aligned} \dot{V}(E) &= \sum_{i=1}^{N} \dot{V}_i(E^i) \\ &= \sum_{i=1}^{N}\Big[-E^{i\top} Q_i E^i + 2E^{i\top} P_i B \\ &\qquad \times \left(g^i + \phi^i(E) + \delta^i(E)\right)\Big] \\ &\leq \sum_{i=1}^{N}\Big[-c_1^i \left\|E^i\right\|^2 + c_2^i \left\|E^i\right\| \\ &\qquad + \left\|E^i\right\| \sum_{j=1}^{N}\left(a^{ij}\left\|E^j\right\|\right)\Big] \end{aligned} \tag{19}$$

with c_1^i, c_2^i and a^{ij} constants and

$$\begin{aligned} c_1^i &= \lambda_{\min}(Q_i)\left(1 - \frac{2\lambda_{\max}(P_i)}{\lambda_{\min}(Q_i)}\left(k_p^i D_{p_1}^i + k_v^i D_{v_1}^i\right)\right) \\ c_2^i &= 2\lambda_{\max}(P_i) \\ &\quad \times \Bigg(k_p^i D_{p_2}^i + k_v^i D_{v_2}^i + k_f^i D_f^i + \left\|k_f^i R^i - \tilde{R}\right\| \\ &\quad + \frac{1}{N}\sum_{j=1}^{N} k_p^j D_{p_2}^j + \frac{1}{N}\sum_{j=1}^{N} k_v^j D_{v_2}^j \\ &\quad + \frac{1}{N}\sum_{j=1}^{N} k_f^j D_f^j + \Delta^i\Bigg) \\ a^{ij} &= \frac{2}{N}\lambda_{\max}(P_i)\left(k_p^j D_{p_1}^j + k_v^j D_{v_1}^j + \sqrt{\Delta k_p^{j\,2} + \Delta k_v^{j\,2}}\right). \end{aligned}$$

Obviously, $c_2^i > 0$, $a^{ij} > 0$, and if we have

$$k_p^i D_{p_1}^i + k_v^i D_{v_1}^i < \frac{1}{\beta_0^i} \tag{20}$$

where

$$\beta_0^i = \frac{2\lambda_{\max}(P_i)}{\lambda_{\min}(Q_i)}$$

then $c_1^i > 0$.

Before we proceed, note that in (20), we want β_0^i to be as small as possible so that the system may tolerate noise with the largest possible bounds ($D_{p_1}^i$ and $D_{v_1}^i$) while keeping stability. Notice that we can influence the size of the β_0^i by the choice of $Q_i > 0$. In fact, we can show that $\min_{Q_i} \beta_0^i = \beta_1^i$. To see this, note that from [19], β_0^i is minimized by letting $Q_i = k_q^i I$ with $k_q^i > 0$ a free parameter, and it can be proven that

$$\min_{Q_i} \beta_0^i = \left.\frac{2\lambda_{\max}(P_i)}{\lambda_{\min}(Q_i)}\right|_{Q_i = k_q^i I} = \frac{k_q^i \beta_1^i}{k_q^i} = \beta_1^i. \tag{21}$$

Thus, from (20) and (21), if (14) holds, then by choosing $Q_i = k_q^i I$, the corresponding constant c_1^i for (19) is positive.

Now, for simplicity, we choose $Q_i = I$ for all i (so $k_q^i = 1$) and use (21), so c_1^i, c_2^i and a^{ij} of (19) are simplified to become

$$c_1^i = 1 - \beta_1^i\left(k_p^i D_{p_1}^i + k_v^i D_{v_1}^i\right) \tag{22}$$

$$\begin{aligned} c_2^i &= \beta_1^i \Bigg(k_p^i D_{p_2}^i + k_v^i D_{v_2}^i + k_f^i D_f^i + \left\|k_f^i R^i - \tilde{R}\right\| \\ &\quad + \frac{1}{N}\sum_{j=1}^{N} k_p^j D_{p_2}^j + \frac{1}{N}\sum_{j=1}^{N} k_v^j D_{v_2}^j \\ &\quad + \frac{1}{N}\sum_{j=1}^{N} k_f^j D_f^j + \Delta^i\Bigg) \end{aligned} \tag{23}$$

$$a^{ij} = \frac{\beta_1^i}{N}\left(k_p^j D_{p_1}^j + k_v^j D_{v_1}^j + \sqrt{\Delta k_p^{j\,2} + \Delta k_v^{j\,2}}\right). \tag{24}$$

Now, return to (19) and note that for any θ^i, $0 < \theta^i < 1$

$$\begin{aligned} &-c_1^i\left\|E^i\right\|^2 + c_2^i\left\|E^i\right\| \\ &\quad = -(1-\theta^i)c_1^i\left\|E^i\right\|^2 - \theta^i c_1^i\left\|E^i\right\|^2 + c_2^i\left\|E^i\right\| \\ &\quad \leq -(1-\theta^i)c_1^i\left\|E^i\right\|^2, \ \forall\ \left\|E^i\right\| \geq r^i \\ &\quad = \sigma^i\left\|E^i\right\|^2 \end{aligned} \tag{25}$$

where $r^i = c_2^i/\theta^i c_1^i$ and $\sigma^i = -(1-\theta^i)c_1^i < 0$. This implies that as long as $\|E^i\| \geq r^i$, the first two terms in (19) combined will give a negative contribution to $\dot{V}(E)$.

Next, we seek conditions under which $\dot{V}(E) < 0$. To do this, we consider the third term in (19) and combine it with the above results. First, note that the third term in (19) can be overbounded by replacing a^{ij} by a^{i^*j} where

$$a^{i^*j} = \max_{1\leq i\leq N} a^{ij} \tag{26}$$

which were defined in the statement of the theorem via (24). Next, we consider the general situation where some of the E^i are such that $\|E^i\| < r^i$ and others are not. Accordingly, define sets

$$\begin{aligned} \Pi_O &= \left\{i : \left\|E^i\right\| \geq r^i,\ i \in 1, \ldots, N\right\} \\ &= \left\{i_O^1,\ i_O^2, \ldots,\ i_O^{N_O}\right\} \end{aligned}$$

and

$$\begin{aligned} \Pi_I &= \left\{i : \left\|E^i\right\| < r^i,\ i \in 1, \ldots, N\right\} \\ &= \left\{i_I^1,\ i_I^2, \ldots,\ i_I^{N_I}\right\} \end{aligned}$$

where N_O and N_I are the size of Π_O and Π_I, respectively, and $N_O + N_I = N$. Also, $\Pi_O \bigcup \Pi_I = \{1, \ldots, N\}$ and $\Pi_O \bigcap \Pi_I = \phi$. Of course, we do not know the explicit sets Π_O and Π_I; all we know is that they exist. The explicit values in the sets clearly depend on time but we will allow that time to be arbitrary so the analysis below will be for all t. For now, we assume $N_O > 0$, that is, the set Π_O is nonempty. We will later discuss the $N_O = 0$ case. Then using analysis ideas from the theory of stability of interconnected systems [20] and using (19), (25), and (26), we have

$$\dot{V}(E) \leq \sum_{i\in\Pi_O} \sigma^i \left\|E^i\right\|^2 + \sum_{i\in\Pi_O} \left(\left\|E^i\right\| \sum_{j\in\Pi_O} a^{i^*j} \left\|E^j\right\|\right) + \sum_{i\in\Pi_O} \left(K_1 + K_3 a^{i^*i}\right) \left\|E^i\right\| + K_2 + K_4$$

where we use the fact that for each fixed N_O, there exist positive constants $K_1(N_O)$, $K_2(N_O)$, $K_3(N_O)$ and $K_4(N_O)$ such that

$$K_1(N_O) \geq \sum_{j\in\Pi_I} a^{i^*j} \left\|E^j\right\|$$
$$K_2(N_O) \geq \sum_{i\in\Pi_I} \left(-c_1^i \left\|E^i\right\|^2 + c_2^i \left\|E^i\right\|\right)$$
$$K_3(N_O) \geq \sum_{i\in\Pi_I} \left\|E^i\right\|$$
$$K_4(N_O) \geq \sum_{i\in\Pi_I} \left(\left\|E^i\right\| \sum_{j\in\Pi_I} a^{i^*j} \left\|E^j\right\|\right). \quad (27)$$

Let $w^\top = \left[\|E^{i_O^1}\|, \|E^{i_O^2}\|, \ldots, \|E^{i_O^{N_O}}\|\right]$ (the composition of this vector can be different at different times) and the $N_O \times N_O$ matrix $S = [s_{jn}]$ be specified by

$$s_{jn} = \begin{cases} -(\sigma^{i_O^j} + a^{i^* i_O^j}), & j = n \\ -a^{i^* i_O^j}, & j \neq n \end{cases} \quad (28)$$

so we have

$$\dot{V}(E) \leq -w^\top S w + \sum_{i\in\Pi_O} \left(K_1 + K_3 a^{i^*i}\right) \left\|E^i\right\| + K_2 + K_4.$$

For now, assume that $S > 0$ in the above equation and, thus, $\lambda_{\min}(S) > 0$. We then have

$$\dot{V}(E) \leq -\lambda_{\min}(S) \sum_{i\in\Pi_O} \left\|E^i\right\|^2 + \sum_{i\in\Pi_O} \left(K_1 + K_3 a^{i^*i}\right) \left\|E^i\right\| + K_2 + K_4. \quad (29)$$

So, when the $\left\|E^i\right\|$ for $i \in \Pi_O$ are sufficiently large, the sign of $\dot{V}(E)$ is determined by $-\lambda_{\min}(S) \sum_{i\in\Pi_O} \left\|E^i\right\|^2$ and $\dot{V}(E) < 0$. This analysis is valid for any value of N_O, $1 \leq N_O \leq N$; hence for any $N_O \neq 0$ the system is uniformly ultimately bounded if $S > 0$, so we seek to prove that next.

A necessary and sufficient condition for $S > 0$ is that its successive principal minors are all positive. Define $|S_m|$ as the determinants of the principal minors of S, $m = 1, \ldots, N_O$. Then, we can show that

$$|S_m| = \left(1 + \sum_{j=1}^{m} \frac{a^{i^* i_O^j}}{\sigma^{i_O^j}}\right) \prod_{k=1}^{m} \left(-\sigma^{i_O^k}\right).$$

Since $-\sigma^{i_O^k} > 0$ for $k = 1, \ldots, m$, to have all the previous determinants positive, we need

$$\sum_{j=1}^{m} \frac{a^{i^* i_O^j}}{\sigma^{i_O^j}} > -1$$

that is

$$\sum_{j=1}^{m} \frac{\beta_1^{i^*} \left(k_p^{i_O^j} D_{p_1}^{i_O^j} + k_v^{i_O^j} D_{v_1}^{i_O^j} + \sqrt{\Delta k_p^{i_O^j\,2} + \Delta k_v^{i_O^j\,2}}\right)}{N\left(1 - \theta^{i_O^j}\right)\left(1 - \beta_1^{i_O^j}\left(k_p^{i_O^j} D_{p_1}^{i_O^j} + k_v^{i_O^j} D_{v_1}^{i_O^j}\right)\right)} < 1$$

for all $m = 1, \ldots, N_O$. Since $1 \leq m \leq N_O \leq N$, the equation above is satisfied when (15) is satisfied and thus, $S > 0$ for all $N_O \neq 0$. Hence, when $\|E^i\|$ is sufficiently large, $\dot{V}(E) < 0$ and the uniform ultimate boundedness of the trajectories of the error system is achieved.

To complete the proof, we need to consider the case when $N_O = 0$. Note that when $N_O = 0$, $\|E^i\| < r^i$ for all i. If we have $N_O = 0$ persistently, then we could simply take $\max_i r^i$ as the uniform ultimate bound. If otherwise, at certain moment the system changes such that some $\|E^i\| \geq \max_i r^i$, then we have $N_O \geq 1$ immediately, then all the analysis above, which holds for any $1 \leq N_O \leq N$, applies. Thus, in either case we obtain the uniform ultimate boundedness. This concludes the proof. ■

Remark 1: Uniform ultimate boundedness is obtained when (14) and (15) are satisfied. Note that these conditions do not depend on k_r^i and r_s^i; these two parameters can affect the size of the ultimate bound but it is the attraction gains k_p^i and k_v^i and damping gain k that determine if boundedness can be achieved for given parameters that quantify the size of the noise. The conditions also do not depend on $D_{p_2}^i$ and $D_{v_2}^i$, but these too will affect the size of the ultimate bound. The conditions do not depend on k_f^i, R^i and D_f^i since our error system quantifies swarm cohesiveness, not how well the resource profile is followed.

Remark 2: From (13), if both k_p^i and k_v^i are fixed, when k is sufficiently large, increasing k will increase β_1^i, which means $D_{p_1}^i$ and $D_{v_1}^i$ have to be decreased to satisfy (14). This means that although we may expect a large k to dampen the error system faster, it could make the system more vulnerable to noise.

Remark 3: Note that when all other parameters are fixed, β_1^i goes to infinity when k_p^i either goes to infinity or approaches zero. Thus, when k_p^i is the only free parameter, there exists some upper bound for $D_{p_1}^i$ beyond which (14) can never hold whatever k_p^i is. This is because when $D_{p_1}^i$ is large enough, k_p^i has to be sufficiently small to decrease the product $k_p^i D_{p_1}^i$ in (14), while the β_1^i corresponding to this sufficiently-small k_p^i will be so large that (14) cannot be satisfied. Basically this means that if $D_{p_1}^i$ is too large and leads to potential instability, it cannot be remedied by merely tuning k_p^i. In comparison, if k_v^i is a free parameter with other parameters fixed and $D_{p_1}^i$ sufficiently small, then for any arbitrarily large $D_{v_1}^i$, we can always find some k_v^i such that (14) still holds. This is because k_v^i and k always appear together in β_1^i and, thus, for any large $D_{v_1}^i$, we are free to decrease k_v^i such that the product of $k_v^i D_{v_1}^i$ is small.

Remark 4: We can see that the smaller $\Delta k_p^{j\,2}$ $(\Delta k_v^{j\,2})$ is, meaning k_p^i (k_v^i) and k_p^j (k_v^j) are closer to each other for all

i and j, the easier it is to meet the condition specified by (15). This means better approximations of the agent parameters may facilitate the boundedness of the error system. In fact when all agents are identical, the sufficient condition (15) can be immediately simplified to

$$\frac{\beta_1(k_p D_{p_1} + k_v D_{v_1})}{(1-\theta)(1-\beta_1(k_p D_{p_1} + k_v D_{v_1}))} < 1$$

by letting $k_p^i = k_p$, $k_v^i = k_v$, $\beta_1^i = \beta_1$, $D_{p_1}^i = D_{p_1}$ and $D_{v_1}^i = D_{v_1}$ for all i. Note the term $\sqrt{\Delta k_p^{i\,2} + \Delta k_v^{i\,2}}$ is zero now since $k_p^i = \bar{k}_p$ and $k_v^i = \bar{k}_v$ for all i. Furthermore, when $D_{p_1}^i = D_{v_1}^i = 0$ for all i, the conditions (14) and (15) will always hold. This means when agents are identical and noise is constant or with constant bound, the trajectories of the error system are always uniformly ultimately bounded. Also, note that since the agents are in general *not* identical and have different parameters, the conditions stated by the theorem are actually quite conservative.

Remark 5: Although we start with plane profiles, the nutrient profile can actually be extended to more general cases. In fact any profile that is smooth and has finite slope at all points, provided the upper bound of its slope is known, can be fit into the framework. To see this, denote the slope upper bound by $\nabla J_{\max}^i$ for agent i, and replace all "R^i" in the system with "$\nabla J_{\max}^i$," then all the proofs follow.

Remark 6: Note that with a repulsion term of the form defined in (2), collision avoidance is not guaranteed. Theorem 1 does not say anything about collision avoidance either. However, as we have found that larger k_r^i and r_s^i, meaning stronger repulsion effect, will result in larger swarm size, we expect they may also help reduce collisions between the agents. See additional discussions on this point in Section V-A. Moreover, as in [12], [16] it is possible to extend the results of this paper to consider a "hard repel" case by using a different form for the repel term.

2) Ultimate Bound on Inter-Agent Trajectories: So far, we have shown that the swarm error system is uniformly ultimately bounded when certain conditions are satisfied. We have shown that the bound exists but have not specified it. If we define the bound as $R_b > 0$, then the set

$$\Omega_c = \{E : \|E^i\| \le R_b, i = 1, \ldots, N\}$$

is attractive and compact. One such bound is given by the Corollary that follows. Before we state the Corollary, some new notation needs to be introduced. Notice that for each given N_O, $1 \le N_O \le N$, the set Π_O can have $N_{N_O} = C_N^{N_O}$ types of compositions, where $C_N^{N_O}$ is the number of combinations of choosing N_O members from a set with N members. (Note that the special case of $N_O = 0$ will be considered separately at the end of the proof for the Corollary.) Let $\Pi_O^{(k)}$ be the set Π_O corresponding to the kth composition, $k = 1, \ldots, N_{N_O}$. Let the $N_O \times N_O$ matrix $S^{(k)}$ be specified in the same way as S, defined in (28), but corresponding to the kth composition, $k = 1, \ldots, N_{N_O}$. Define for $k = 1, \ldots, N_{N_O}$

$$\begin{aligned} a_d^k(N_O) &= \lambda_{\min}(S_{N_O \times N_O}^{(k)}) \\ b_d(N_O) &= K_1(N_O) + K_3(N_O)\hat{a} \\ c_d(N_O) &= K_2(N_O) + K_4(N_O) \end{aligned}$$

where K_1, K_2, K_3 and K_4 are defined in (27), and

$$\hat{a} = \max_{1\le i\le N, 1\le j\le N} a^{ij} = \max_{1\le j\le N} a^{i^* j}$$

with $a^{i^* j}$ defined in (26). Note that in Theorem 1 we do *not* highlight the difference between compositions because it does not matter, while in the proof for the Corollary below it will make things more clear to do so. Also note that b_d and c_d are not affected by the composition because we may choose K_1, K_2, K_3, and K_4 in such a way that (27) always holds for *any* composition with $0 \le N_I \le N-1$ (and, thus, $1 \le N_O \le N$). In the following Corollary and proof, all notation is the same as those in Theorem 1 unless otherwise specified.

Corollary 1: Define $r^* = \max_{1\le i\le N} r^i$, with r^i defined in (25) via some set of θ^i that satisfy (15). When the conditions in Theorem 1 are all satisfied, there exists some constant $0 < \theta_d < 1$ such that the uniform ultimate bound of the trajectories of the error system is

$$R_b = \max\{r_b, r^*\}$$

where

$$r_b = \frac{b_d^* + \sqrt{N b_d^{*2} + 4a_d^* c_d^*}}{2a_d^*\theta_d} \tag{30}$$

with a_d^*, b_d^*, and c_d^* are all constants and

$$\begin{aligned} a_d^* &= \min_{k,N_O} a_d^k(N_O) \\ b_d^* &= \max_{N_O} b_d(N_O) \\ c_d^* &= \max_{N_O} c_d(N_O) \end{aligned}$$

for $N_O = 1, \ldots, N$ and $k = 1, \ldots, N_{N_O}$.

Proof: Note that when the conditions of Theorem 1 are all satisfied, a set of constants θ^i exists and can be found. Also recall that both c_1^i and c_2^i are constants, then $r^i = c_2^i/\theta^i c_1^i$ and $\sigma^i = -(1-\theta^i)c_1^i$, defined in (25), are constants for all i and can be found. Thus, numeric values of a_d^*, b_d^*, and c_d^* can be found in terms of known parameters. Now we will first show that this Corollary applies to a fixed $N_O \ne 0$ with a particular composition k.

With $V(E)$ defined in Theorem 1, for $N_O \ne 0$ and $\Pi_O^{(k)}$, from (29), we have

$$\begin{aligned} \dot{V}(E) &\le -a_d^k \sum_{i\in\Pi_O^{(k)}} \|E^i\|^2 + b_d \sum_{i\in\Pi_O^{(k)}} \|E^i\| + c_d \\ &= \sum_{i\in\Pi_O^{(k)}} \underbrace{\left[-a_d^k\left(\|E^i\| - \frac{b_d}{2a_d^k}\right)^2 + \frac{b_d^2}{4a_d^k}\right]}_{F^i} + c_d \end{aligned} \tag{31}$$

with a_d^k, b_d and c_d all positive constants. Notice we want to find a bound r_b' such that $\dot{V}(E) < 0$ so long as there exist some $\|E^i\| > r_b'$, $i \in \Pi_O^{(k)}$. Before we start to solve for this r_b', note that F^i in (31) can be visualized by Fig. 1, where F^i is a parabolic function with respect to $\|E^i\|$ and crosses $\|E^i\|$ axis at two points r_A and r_B, respectively. To have $\dot{V}(E) < 0$, one possibility is such that $\|E^i\| > r_B$ and thus, $F^i < 0$ for all $i \in \Pi_O^{(k)}$. (Note that due to the nature of our problem, we do not consider the case of $\|E^i\| < r_A$, though this also results in $F^i < 0$.) We call this situation the "best situation." While in a more general case, we have some $\|E^i\| > r_B$ while all the other $\|E^i\| \le r_B$, $i \in \Pi_O^{(k)}$ and we call such a situation the "normal situation." For the best situation case, each $\|E^i\|$ just needs to be *a little* bigger than r_B to achieve $\dot{V}(E) < 0$, which means r_b' is just a little bigger than r_B. In comparison, to get $\dot{V}(E) < 0$ for the normal situation, generally it means those $\|E^i\|$ that satisfy $\|E^i\| > r_B$ have to be further to the right on $\|E^i\|$ axis (i.e., *much* bigger than r_B) to counteract the "positive" effects brought in by those $\|E^i\|$ with $\|E^i\| \le r_B$, meaning r_b' needs to be much bigger than r_B. With this idea, we can see that the worst r_b' happens when there is only *one* F^i, call it $i = i'$, free to change while all the other F^i, with $i = i_O^1, \ldots, i_O^{N_O}$ and $i \ne i'$, are fixed at their respective maximum (or most "positive" value). Basically this depicts a situation when there is only one agent having its norm of error $\|E^{i'}\|$ slide along $\|E^i\|$ axis (to the right) to bring $\dot{V}(E)$ down to negative value while all other agents in $\Pi_O^{(k)}$ stay in positions as bad as they can (in the sense of keeping stability).

From (31), we can see that each F^i achieves its maximum of $b_d^2/4a_d^k$ with $\|E^i\| = b_d/2a_d^k$. Then, based on the previous analysis, we can solve for the r_b' by letting all $F^i = b_d^2/4a_d^k$ except for $i = i'$. From (31), for some constant $0 < \theta_d < 1$ we have

$$\begin{aligned}\dot{V}(E) &\le \left[-a_d^k\left(\left\|E^{i'}\right\| - \frac{b_d}{2a_d^k}\right)^2 + \frac{b_d^2}{4a_d^k}\right] \\ &\quad + \sum_{i\in\Pi_O^{(k)}, i\ne i'} \frac{b_d^2}{4a_d^k} + c_d \\ &\le -a_d^k(1-\theta_d)\left\|E^{i'}\right\|^2 \qquad \forall \left\|E^{i'}\right\| \ge r_b'\end{aligned}$$

where

$$r_b' = \frac{b_d + \sqrt{N_O b_d^2 + 4a_d^k c_d}}{2a_d^k \theta_d}.$$

Note that in setting $\|E^i\| = b_d/2a_d^k$ to get $F^i = b_d^2/4a_d^k$, we may violate the prerequisite of $i \in \Pi_O^{(k)}$ since it may happen that $r^i > b_d/2a_d^k$ for some $i \in \Pi_O^{(k)}$. However, this violation only adds more conservativeness to the resultant r_b' and does not nullify the fact that r_b' is a valid bound. Specifically, when $r^i > b_d/2a_d^k$ for some $i \in \Pi_O^{(k)}$, the corresponding F^i become "less" positive, as seen from Fig. 1, and thus, the actual bound will be smaller than the r_b' obtained above. Hence, r_b' is still a valid (upper) bound.

Since $r_b \ge r_b'$ with r_b defined in (30), and the positive constant $a_d^* \le a_d^k$, we have $\dot{V}(E) \le -a_d^*(1-\theta_d)\left\|E^i\right\|^2$ when $\left\|E^i\right\| \ge r_b$. Now, note that the choice of N_O and composition k in the above proof are in fact arbitrary (except that $N_O \ne 0$) and can be time-varying. So we actually have $R_b \ge r_b \ge r_b'$ for any $1 \le N_O \le N$ and any composition at any time t. That is, the proof above is actually valid for the general case when N_O ($N_O \ne 0$) and the composition are time varying.

To complete the proof, we need to show that R_b is also a valid bound for the case of $N_O = 0$, i.e., an empty set Π_O. Notice

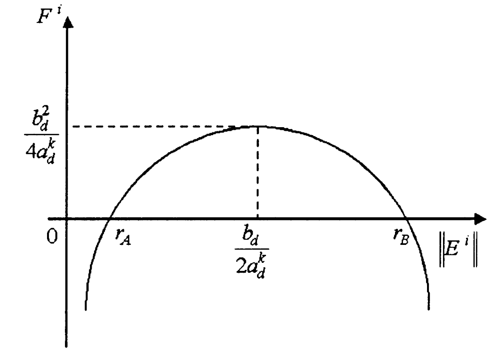


Fig. 1. F^i vs $\|E^i\|$.

that $N_O = 0$ means $\|E^i\| < r^i$ for $i = 1, \ldots, N$. Also notice that $r^i \le \max_{1\le i\le N} r^i \le R_b$ for all i. Then, by definition, as long as $N_O = 0$, the trajectories of the swarm error system stay within the bound R_b. This concludes the proof. ■

Remark 7: The value of R_b is affected by two components: r_b and r^*. We will discuss r_b first. It is easy to see from (30) that increasing a_d^* helps to decrease r_b. Notice a_d^* is a function including many parameters, so it is difficult to provide clear relationships to their effects on r_b. But we may get some ideas based on intuition. Note that a_d^* is related to the minimum eigenvalue of matrix S defined in (28), where the components $-(\sigma^i + a^{i^*i})$, $i \in 1, \ldots, N$, on the main diagonal reflect the stabilizing effect of the isolated parts of the composite system, while the cross terms $-a^{i^*i}$ reflect the destabilizing effect of the interconnection parts of the composite system. (By abuse of notation, above we use $-(\sigma^i + a^{i^*i})$ instead of $-(\sigma^{i_O^j} + a^{i^*i_O^j})$ as in (28). This is because we are discussing the general case and each i_O^j can be any value from 1 to N. Similarly, in this Remark and the next Remark, we do not tell the difference of a^{i^*i} and a^{i^*j}, since both i and j range from 1 to N.) So, the larger *magnitude* those main diagonal components have (or relatively, the smaller *magnitude* those cross terms have), the "more stable" the composite system is and, thus, we may expect the larger a_d^* is. Based on this analysis and (22), since smaller $D_{p_1}^i$ and $D_{v_1}^i$ gives larger c_1^i and, thus, larger σ^i in magnitude (implying "better stability"), it may render larger a_d^* and thus, smaller r_b. When σ^i are relatively big, from (24), we may deduce that smaller N and thus, larger a^{i^*i} (implying "worse stability") may lead to smaller a_d^* and thus, larger r_b.

Remark 8: Note that b_d^* is affected by a^{i^*j} via $\hat{a}$, $j \in 1, \ldots, N$. Larger a^{i^*j} may lead to larger b_d^* and thus, a larger bound r_b. This is consistent with the analysis in the previous remark. Similar conclusions may be drawn by inspecting c_d^* since it includes K_4, which is affected by a^{i^*j}. It is interesting to note that from (30), smaller N means smaller r_b, while in the previous remark we mention that smaller N may lead to larger a^{i^*j} and, thus, larger r_b. These seemingly contradictive conclusions in fact make sense intuitively. Too large N does not help reducing r_b because with each agent desiring to keep certain distance from others, large N means large swarm radius. Too small N does not always help reducing r_b because the effect of noise becomes more significant when N is small. In the other words, with smaller N, the swarm cannot "average" out the noise and thus, the bound on the trajectories is not reduced. This "noise-averaging" idea will become more clear in Section III-C, when we deal with identical agents, and later in other simulations.

Remark 9: The value of R_b is also affected by r^*. Note that r^* is determined by c_1^i and c_2^i for all i. Specifically, smaller c_2^i and larger c_1^i are helpful in decreasing r^*. Then from (22) and (23), we can see that all the noise bounds ($D_{p_1}^i, D_{v_1}^i, D_{p_2}^i, D_{v_2}^i$, and D_f^i) affect r^*. Smaller noise bounds help decrease r^* and thus, may decrease R_b. So do smaller k_f^i, k_r^i and r_s^i. All these are consistent with the previous remarks.

Remark 10: Similar to Theorem 1, lots of conservativeness is introduced into the deduction of Corollary 1 . One example is, K_1, K_2, K_4, and thus, b_d^* and c_d^*, are actually functions of N_O. When N_O increases, b_d^* and c_d^* will generally decrease. This fact is not considered in the above deduction because of the complexity that originated in the use of both heterogeneous swarm agents and resource profiles in the environment.

C. Special Case: Identical Agents

Here, we will study the stability of the system when all the agents are identical (i.e., with $k_p^i = k_p$, $k_v^i = k_v$, $k_r^i = k_r$, $r_s^i = r_s$, and $k_f^i = k_f$, for all i), but with different types of noise and nutrient profiles. Equation (4) becomes

$$\begin{aligned} u^i =& -M_i k_p \hat{e}_p^i - M_i k_v \hat{e}_v^i - M_i k v^i \\ &+ M_i k_r \sum_{j=1, j\neq i}^{N} \exp\left(\frac{-\frac{1}{2}\left\|\hat{e}_p^i - \hat{e}_p^j\right\|^2}{{r_s}^2}\right)\left(\hat{e}_p^i - \hat{e}_p^j\right) \\ &- M_i k_f \left(R^i - d_f^i\right). \end{aligned} \tag{32}$$

Also, $A_i = A$ in (11) for all i. Note that with all agents being identical, we have $\Delta k_p^i = 0$ and $\Delta k_v^i = 0$ for all i. Also

$$\frac{1}{N}\sum_{l=1}^{N} k_r \sum_{j=1, j\neq l}^{N} \exp\left(\frac{-\frac{1}{2}\left\|\hat{e}_p^l - \hat{e}_p^j\right\|^2}{{r_s^l}^2}\right)\left(\hat{e}_p^l - \hat{e}_p^j\right) = 0.$$

Let $\bar{d}_p = 1/N\sum_{i=1}^N d_p^i$, $\bar{d}_v = 1/N\sum_{i=1}^N d_v^i$, $\bar{d}_f = 1/N\sum_{i=1}^N d_f^i$, and $\bar{R} = 1/N\sum_{i=1}^N R^i$. Then, (6) can be simplified to

$$\dot{\bar{v}} = -k\bar{v} + \underbrace{k_p\bar{d}_p + k_v\bar{d}_v + k_f\bar{d}_f - k_f\bar{R}}_{z(t)} \tag{33}$$

and

$$\dot{e}_v^i = -k_p e_p^i - (k_v + k)\, e_v^i + g^i + \phi(E) + \delta^i(E) \tag{34}$$

where g^i, $\phi(E)$, and $\delta^i(E)$ are, respectively

$$g^i = k_p d_p^i + k_v d_v^i + k_f d_f^i - k_f R^i \tag{35}$$

$$\phi(E) = -k_p\bar{d}_p - k_v\bar{d}_v - k_f\bar{d}_f + k_f\bar{R} \tag{36}$$

$$\delta^i(E) = k_r \sum_{j=1, j\neq i}^{N} \exp\left(\frac{-\frac{1}{2}\left\|\hat{e}_p^i - \hat{e}_p^j\right\|^2}{{r_s}^2}\right)\left(\hat{e}_p^i - \hat{e}_p^j\right). \tag{37}$$

Using the idea of deriving (18), we have

$$\left\|\delta^i(E)\right\| \le k_r r_s (N-1)\exp\left(-\frac{1}{2}\right). \tag{38}$$

1) Noise With Constant Bounds: In this case, we assume that $d_p^i(t)$ and $d_v^i(t)$ are sufficiently smooth and bounded by some constants for all i

$$\begin{aligned} \|d_p^i\| &\le D_p \\ \|d_v^i\| &\le D_v \end{aligned} \tag{39}$$

where $D_p \ge 0$ and $D_v \ge 0$ are known constants. The sensing error on the gradient of the nutrient profile is assumed to be sufficiently smooth and bounded by known constant $D_f \ge 0$ such that for all i

$$\|d_f^i\| \le D_f. \tag{40}$$

Theorem 2: Consider the swarm described by the model in (3) with control input u^i given as in (32). Assume that the nutrient profile for each agent is a plane defined by $\nabla J_p^i(x) = R^i$. Also, assume the noise satisfies (39) and (40). Let $\|R^*\| = \max_i \left\|R^i - \bar{R}\right\|$. Then, the trajectories of the swarm error system are uniformly ultimately bounded, and E^i for all i will converge to the set Ω_b, where

$$\Omega_b = \left\{E : \left\|E^i\right\| \le \beta_1\beta_2,\ i = 1, 2, \ldots, N\right\} \tag{41}$$

is attractive and compact, with

$$\beta_1 = \frac{(k_p+1)^2 + (k_v+k)^2}{2k_p(k_v+k)} + \sqrt{\left(\frac{k_p^2 + (k_v+k)^2 - 1}{2k_p(k_v+k)}\right)^2 + \frac{1}{k_p^2}}$$

and

$$\beta_2 = 2k_pD_p + 2k_vD_v + 2k_fD_f + k_f\|R^*\| + k_r r_s (N-1)\exp\left(-\frac{1}{2}\right).$$

Moreover, there exists some finite T and constant $0 < \theta < 1$ such that

$$\|\bar{v}(t)\| \le \exp\left[-(1-\theta)kt\right]\|\bar{v}(0)\|,\ \forall\, 0 \le t < T$$

and

$$\|\bar{v}(t)\| \le \frac{\delta}{k\theta} \qquad \forall\, t \ge T$$

with $\delta = k_pD_p + k_vD_v + k_fD_f + k_f\|\bar{R}\|$.

Proof: To find the set Ω_b, from (17) and (34) to (38), we have

$$\dot{V}_i \le -\lambda_{\min}(Q_i)\left\|E^i\right\|\left(\left\|E^i\right\| - 2\frac{\lambda_{\max}(P_i)}{\lambda_{\min}(Q_i)}\beta_2\right).$$

Following the idea in Theorem 1, we have by letting $Q_i = I$, β_1 as the counterpart of β_1^i in Theorem 1. So we have

$$\dot{V}_i \le -\left\|E^i\right\|\left(\left\|E^i\right\| - \beta_1\beta_2\right). \tag{42}$$

That is, $\dot{V}_i < 0$ if $\left\|E^i\right\| > \beta_1\beta_2$. So, the set

$$\Omega_b = \left\{E : \left\|E^i\right\| \le \beta_1\beta_2,\ i = 1, 2, \ldots, N\right\}$$

is attractive and compact. Also, we know that within a finite amount of time, $E^i \to \Omega_b$. This means that we can guarantee that if the swarm is not cohesive, it will seek to be cohesive, but only if it is a certain distance from cohesiveness as indicated by (42).

To study the boundedness of $\bar{v}(t)$, choose a Lyapunov function

$$V_{\bar{v}} = \frac{1}{2}\bar{v}^\top \bar{v}$$

defined on $D = \{\bar{v} \in \Re^n \mid \|\bar{v}\| < r_v\}$ for some $r_v > 0$, and we have

$$\dot{V}_{\bar{v}} = \bar{v}^\top \dot{\bar{v}} = -k\bar{v}^\top \bar{v} + \bar{v}^\top z(t)$$

with $z(t)$ defined in (33). Since $\left\|d_p^j\right\| \leq D_p$ for all j, we have $\left\|\bar{d}_p\right\| = \left\|1/N\sum_{j=1}^N d_p^j\right\| \leq D_p$. Similarly, $\left\|\bar{d}_v\right\| \leq D_v$ and $\left\|\bar{d}_f\right\| \leq D_f$. Thus, we have

$$\|z(t)\| \leq \left\|k_p\bar{d}_p\right\| + \left\|k_p\bar{d}_v\right\| + \left\|k_f\bar{d}_f\right\| + \left\|k_f\bar{R}\right\| \leq \delta.$$

If $\delta < k\theta r_v$ for all $t \geq 0$, all $\bar{v} \in D$ and some positive constant $\theta < 1$, then it can be proven that for all $\|\bar{v}(0)\| < r_v$ and some finite T we have

$$\|\bar{v}(t)\| \leq \exp\left[-(1-\theta)kt\right]\|\bar{v}(0)\|, \ \forall\, 0 \leq t < T$$

and

$$\|\bar{v}(t)\| \leq \frac{\delta}{k\theta} \qquad \forall\, t \geq T.$$

Since this holds globally we can take $r_v \to \infty$ so these equations hold for all $\bar{v}(0)$. ■

Remark 11: The size of Ω_b in (41), which we denote by $|\Omega_b|$, is directly a function of several known parameters. If there are no sensing errors, i.e., $D_p = D_v = D_f = 0$, then Ω_b reduces to the set representing the no-noise case. For fixed values of N, k_p, k_v, k, and k_r if we increase r_s each agent has a larger region from which it will repel its neighbors so $|\Omega_b|$ is larger. For fixed k_r, k_p, k_v, k, and r_s if we let $N \to \infty$, then $|\Omega_b| \to \infty$ as we expect due to the repulsion.

Remark 12: It is interesting to note that in some swarms N is very large and when there is no biasing of sensing errors, we have $\bar{d}_p \approx \bar{d}_v \approx \bar{d}_f \approx 0$. This reduces the bound defined by β_2. Also when $\|R^*\|$ is decreased, implying that R^i is closer to R^j for all i and j, then β_2 is smaller. In the special case when all R^i are the same, we have $\|R^*\| = 0$ and the set $|\Omega_b|$ is minimized with respect to resource profiles. This means when $\|R^*\|$ is large, the agents pursue resource profiles that are far different from each other and the swarm is spread out, while when those profiles are equal to each other, all the agents move along the same profile and smaller swarm size is achieved.

Remark 13: If δ and θ are fixed, with increasing k $\|\bar{v}(t)\|$ decreases faster for $0 \leq t < T$ and has smaller bound for $t \geq T$. If δ gets larger with k and θ fixed, $\|\bar{v}(t)\|$ has larger bound for $t \geq T$; hence if the magnitude of the noise becomes larger this increases δ and hence there can be larger magnitude changes in the ultimate average velocity of the swarm (e.g., the average velocity could oscillate). Note that if in (33) $z(t) \approx 0$ (e.g., due to noise that destroys the directionality of the resource profile R), then the above bound may be reduced but the swarm could be going in the wrong direction.

Remark 14: Regardless of the size of the bound it is interesting to note that while the noise destroys the ability of an individual agent to follow a gradient accurately, the average sensing errors of the group are what changes the direction of the group's movement relative to the direction of the gradient of $J_p(x)$. In some cases when the swarm is large (N big) it can be that $\bar{d}_p \approx \bar{d}_v \approx \bar{d}_f \approx 0$ since the average sensing error is zero and the group will perfectly follow the proper direction for foraging (this may be a reason why for some organizms, large group size is favorable). In the case when $N = 1$ (i.e., single agent), there is no opportunity for a cancellation of the sensor errors; hence an individual may not be able to climb a noisy gradient as easily as a group. This characteristic has been found in biological swarms [13], [14].

Remark 15: Note that there is an intimate relationship between sensor noise and observations of biological swarms (e.g., in bee swarms) that there is a type of "inertia" of a swarm. Note that for large swarms (high N) there can be regions where the average sensor noise is small so that agents in that region move in the right direction. In other regions there may be alignments of the errors and hence the agents may not be all moving in the right direction so they may get close to each other and impede each other's motion, having the effect of slowing down the whole group. With no noise the group inertia effect is not found since each agent is moving in the right direction. The presence of sensor noise generally can make it more difficult to get the group moving in the right (foraging) direction. Large swarms can help move the group in the right direction, but at the expense of possibly slowing their movement initially in a transient period.

2) Constant Errors: In this case, we assume each agent senses the velocity and position of other members and the nutrient profile with some constant errors.

Theorem 3: Consider the swarm described by the model in (3) with control input u^i given as in (32). Assume that the nutrient profile for each agent is a plane defined by $\nabla J_p^i(x) = R^i$. Also, assume the noise d_p^i, d_v^i, and d_f^i are time-invariant for each agent so that $\bar{d}_p = 1/N\sum_{i=1}^N d_p^i$, $\bar{d}_v = 1/N\sum_{i=1}^N d_v^i$, $\bar{d}_f = 1/N\sum_{i=1}^N d_f^i$, and $\bar{R} = 1/N\sum_{i=1}^N R^i$ are constants. Then, the error dynamics of the swarm system are uniformly ultimately bounded and E^i, $i = 1, \ldots, N$, will converge to the attractive and compact set Ω_t defined by

$$\Omega_t = \left\{E : \left\|E^i\right\| \leq \alpha^i\beta_1,\ i = 1, \ldots, N\right\} \tag{43}$$

where β_1 is defined in Theorem 2 and

$$\alpha^i = \left\|k_p\left(d_p^i - \bar{d}_p\right) + k_v\left(d_v^i - \bar{d}_v\right) + k_f\left(d_f^i - \bar{d}_f\right) - k_f\left(R^i - \bar{R}\right)\right\| + k_r r_s(N-1)\exp\left(-\frac{1}{2}\right).$$

Moreover, $e_v^i \to 0$ and

$$v^i(t) \to \frac{k_p\bar{d}_p + k_v\bar{d}_v + k_f\bar{d}_f - k_f\bar{R}}{k} \tag{44}$$

for all i as $t \to \infty$.

Proof: We may obtain the set Ω_t by following the method in Theorem 2 , α^i as the counterpart of β_2 in Theorem 2.

To find the ultimate velocity of each agent in the swarm, we consider Ω_t and a Lyapunov function $V^o(E) = \sum_{i=1}^N V_i^o(E^i)$ with

$$V_i^o(E^i) = \frac{1}{2}k_p\left(e_p^i - \frac{\gamma}{k_p}\right)^\top\left(e_p^i - \frac{\gamma}{k_p}\right) + \frac{1}{2}e_v^{i\top}e_v^i + k_r r_s^2 \sum_{j=1, j\neq i}^N \exp\left(\frac{-\frac{1}{2}\left\|\hat{e}_p^i - \hat{e}_p^j\right\|^2}{r_s^2}\right) \quad (45)$$

where the constant $\gamma = k_p(d_p^i - \bar{d}_p) + k_v(d_v^i - \bar{d}_v) + k_f\left(d_f^i - \bar{d}_f\right) - k_f(R^i - \bar{R})$. Note that this Lyapunov function is not positive definite, but $V_i^o(E^i) > 0$. Here, we think of the swarm moving so as to *minimize* $V^o(E)$ with the i^{th} agent trying to minimize $V_i^o(E^i)$. Agents try to place themselves at positions to reduce the first term in (45), achieve a velocity to reduce the second term, and move to a distance from each other to minimize repulsion quantified in the last term. There is a resulting type of balance that is sought between the conflicting objectives that each of the three terms represent.

Using (34), we have

$$\dot{V}_i^o = -(k_v + k)e_v^{i\top}e_v^i.$$

Hence, $\dot{V}^o = -(k_v + k)\sum_{i=1}^N \|e_v^i\|^2 \leq 0$ on $E \in \Omega$ for any compact set Ω. Choose Ω so it is positively invariant, which is clearly possible, and so $\Omega_e \in \Omega$ where

$$\begin{aligned}\Omega_e &= \{E : \dot{V}^o(E) = 0\} \\ &= \{E : e_v^i = 0, \qquad i = 1, 2, \ldots, N\}.\end{aligned}$$

From LaSalle's Invariance Principle, we know that if $E(0) \in \Omega$ then $E(t)$ will converge to the largest invariant subset of Ω_e. Hence $e_v^i(t) \to 0$ as $t \to \infty$. From (33), we have

$$\bar{v}(t) \to \frac{k_p\bar{d}_p + k_v\bar{d}_v + k_f\bar{d}_f - k_f\bar{R}}{k}$$

as $t \to \infty$ since in this case $z(t)$ is a constant with respect to time. Thus, $v^i(t)$ approaches this value also for all i as $t \to \infty$. ■

Remark 16: From (44), we can see that all agents will ultimately be moving at the same velocity despite the existence of constant errors. Contrast this with the earlier more general cases where it is possible that $\bar{v}$ and v^i ultimately, for example, oscillate. Next, note that even if $R^i = R^j$ for all i and j, the presence of $\bar{d}_p$, $\bar{d}_v$, and $\bar{d}_f$ represent the effects of sensor errors and they can result in the swarm not properly following the direction of the profile even when they all intend to go in the same direction. In the case when we have $\bar{d}_p \approx \bar{d}_v \approx \bar{d}_f \approx 0$ with N large enough, then all those agents will be following the "averaged" profile $-k_f/k\bar{R}$. That is, due to the desire to stay together, they each sacrifice following their own profile and compromise to follow the averaged profile. In the special case when $\bar{R} = 0$ or $R^i = 0$ for all i (no resource profile effect), and no sensor errors, both $\bar{v}(t)$ and $v^i(t)$ will go to zero as $t \to \infty$, representing the aggregation of the group independent of the environment. The size of Ω_t in (43), which we denote by $|\Omega_t|$, is directly a function of several known parameters. All the remarks for the previous case, i.e., noise with constant bounds, apply here.

IV. Stability Analysis of Swarm Trajectory Following

In this section, we briefly analyze the stability of the swarm error system when each agent is trying to track their respective trajectories. This is done by applying the same idea as in Section III to a slightly reformulated system model. Specifically, redefine the errors as

$$\begin{aligned}e_p^i &= x^i - x_d^i \\ e_v^i &= v^i - v_d^i\end{aligned}$$

where x_d^i is a sufficiently smooth desired position trajectory for agent i, $i = 1, \ldots, N$, and $v_d^i = \dot{x}_d^i$. We assume that there exist known bounds for $\dot{x}_d^i$ and $\dot{v}_d^i$ such that

$$\begin{aligned}\left\|\dot{x}_d^i\right\| &\leq D_{x_d^i} \\ \left\|\dot{v}_d^i\right\| &\leq D_{v_d^i}\end{aligned}$$

where $D_{x_d^i}$ and $D_{v_d^i}$ are known positive constants. Assume the nutrient profile for each agent is a plane defined by $\nabla J_p^i(x) = R^i$. Also let $\dot{e}_p^i$, $\dot{e}_v^i$, $\hat{e}_p^i$, $\hat{e}_v^i$, u^i, and E^i be defined in the same form as in Section III-A. Then, the error dynamics of the i^{th} agent is

$$\dot{E}^i = \begin{bmatrix} 0 & I \\ -k_p^i I & -(k_v^i + k)I \end{bmatrix} E^i + \begin{bmatrix} 0 \\ I \end{bmatrix}\left(g^i + \delta^i(E)\right) \quad (46)$$

where

$$g^i = k_p^i d_p^i + k_v^i d_v^i + k_f^i d_f^i - k_f^i R^i \quad (47)$$

$$\delta^i(E) = k_r^i \sum_{j=1, j\neq i}^N \exp\left(\frac{-\frac{1}{2}\left\|\hat{e}_p^i - \hat{e}_p^j\right\|^2}{{r_s^i}^2}\right)\left(\hat{e}_p^i - \hat{e}_p^j\right) - kv_d^i - \dot{v}_d^i. \quad (48)$$

Let $d_f^i(t)$, $d_p^i(t)$, and $d_v^i(t)$ be specified in the same way as in Section III-B. Then, we have the following theorem.

Theorem 4: Consider the swarm described by the model in (3) with control input u^i given in (4). Let β_1^i be defined in Theorem 1 . If for all i we have

$$k_p^i D_{p_1}^i + k_v^i D_{v_1}^i < \frac{1}{\beta_1^i} \quad (49)$$

then the trajectories of the error system, specified by (46), are uniformly ultimately bounded. Furthermore, E^i for all i will converge to an attractive and compact set Ω_f defined as

$$\Omega_f = \left\{E : \left\|E^i\right\| \leq \frac{\tilde{c}_2^i}{\tilde{c}_1^i}, i = 1, \ldots, N\right\} \quad (50)$$

where

$$\tilde{c}_1^i = 1 - \beta_1^i\left(k_p^i D_{p_1}^i + k_v^i D_{v_1}^i\right) \quad (51)$$

$$\tilde{c}_2^i = \beta_1^i\left(k_p^i D_{p_2}^i + k_v^i D_{v_2}^i + k_f^i D_f^i + k_f^i\left\|R^i\right\| + kD_{x_d^i} + D_{v_d^i} + \hat{\delta}^i\right) \quad (52)$$

with $\hat{\delta}^i = k_r^i r_s^i (N-1)\exp(-1/2)$. Moreover, if we have for any i and j

$$\left\|x_d^i - x_d^j\right\| \leq D_x \quad (53)$$

where D_x is a known constant, then the swarm will stay cohesive and

$$\left\|x^i - \bar{x}\right\| \leq \frac{\tilde{c}_2^i}{\tilde{c}_1^i} + \frac{1}{N}\sum_{j=1}^{N}\frac{\tilde{c}_2^j}{\tilde{c}_1^j} + D_x \tag{54}$$

for all i.

Proof: Note that g^i and $\delta^i(E)$, defined in (47) and (48), are bounded by $k_p^i D_{p_2}^i + k_v^i D_{v_2}^i + k_f^i D_f^i + k_f^i \left\|R^i\right\|$ and $kD_{x_d^i} + D_{v_d^i} + \hat{\delta}^i$, respectively. By following exactly the same method in the proof of Theorem 1, we obtain

$$\dot{V}_i(E^i) \leq -\tilde{c}_1^i \left\|E^i\right\|^2 + \tilde{c}_2^i \left\|E^i\right\| \tag{55}$$

and define $V(E) = \sum_{i=1}^{N} V_i(E^i)$ as the Lyapunov function for the whole error system, as specified in the proof of Theorem 1. Then the uniform ultimate boundedness of the error system and the set Ω_f are easily obtained via (55).

When (53) is satisfied, we can show that the cohesiveness of the swarm is conserved. To see this, note that for arbitrary i and j with $i \neq j$

$$\begin{aligned}\left\|x^i - x^j\right\| &= \left\|\left(e_p^i + x_d^i\right) - \left(e_p^j + x_d^j\right)\right\| \\ &\leq \left\|e_p^i - e_p^j\right\| + \left\|x_d^i - x_d^j\right\| \\ &\leq \left\|E^i\right\| + \left\|E^j\right\| + D_x \end{aligned} \tag{56}$$

so, from (50) and (56), we have (54). ■

Remark 17: Comparing (19) with (55), we can see that the latter one does not include any cross term. This is because the errors for the swarm cohesion case are defined as the difference between an agent and the swarm centers ($\bar{x}$ and $\bar{v}$), which are affected by all the agents in the swarm, while the errors for the trajectory following case are defined as the difference between an agent and the given position and velocity trajectories, which are *not* affected by the behaviors of other agents. This absence of cross term significantly simplifies the proof for the theorem.

Remark 18: By comparing Theorem 1 and Theorem 4, we can see that Theorem 4 will hold whenever Theorem 1 holds, as long as $D_{x_d^i}$ and $D_{v_d^i}$ exist. This means cohesion property of a swarm in a certain environment guarantees the stability of that swarm in following *any* bounded trajectory in the same environment.

Remark 19: Similar to Remark 1, when (49) holds, the uniform ultimate boundedness is obtained. This condition only depends on k_p^i, k_v^i, D_{p_1}, and D_{v_1}. Although the remaining parameters, including k_r^i, r_s^i, k_f^i, D_{p_2}, D_{v_2}, D_f, and R^i, do not affect the boundedness of the error system, they do affect the ultimate bound. In the special case when $D_{p_1}^i = D_{v_1}^i = 0$ for all i, (49) always holds and, thus, the swarm error system is always bounded.

Remark 20: Smaller $\left\|R^i\right\|$ may decrease the bound. Smaller magnitude of the position and velocity trajectories also help in decreasing the ultimate bound. Our analysis includes the possibility that the resource profiles indicate that the agents should go in the opposite direction that is indicated by (x_d^i, v_d^i). If there is an alignment between where the resource profiles say to go and the (x_d^i, v_d^i), then the size of the bound decreases.

Remark 21: In the special case when v_d^i and all sensing errors are constant, we have that $e_v^i \to 0$ as $t \to \infty$ for all i, meaning v^i of each agent will be precisely following the given constant velocity trajectory ultimately in such a case. To see this, let $\gamma^i = k_p^i d_p^i + k_v^i d_v^i - kv_d^i + k_f^i d_f^i - k_f^i R^i$ and construct for all i a Lyapunov function

$$\begin{aligned} V_i^o\left(E^i\right) = &\frac{1}{2}k_p^i\left(e_p^i - \frac{\gamma^i}{k_p^i}\right)^\top\left(e_p^i - \frac{\gamma^i}{k_p^i}\right) + \frac{1}{2}e_v^{i\top} e_v^i \\ &+ k_r^i {r_s^i}^2 \sum_{j=1, j\neq i}^{N} \exp\left(\frac{-\frac{1}{2}\left\|\hat{e}_p^i - \hat{e}_p^j\right\|^2}{r_s^2}\right). \end{aligned}$$

Note $\dot{v}_d^i = 0$ when v_d^i is constant. Then by following the method in the proof of Theorem 3, the aforementioned claim holds.

V. Simulations

In this section, we will show some simulation results for both the no-noise and noise cases. Unless otherwise stated, in all the following simulations the parameters, which we refer to as "normal parameters," are: $N = 50$, $k_p^i = k_p = 1$, $k_v^i = k_v = 1$, $k = 0.1$, $k_f^i = k_f = 0.1$, $k_r^i = k_r = 10$, $r_s^i = r_s = 0.1$, and the three dimensional nutrient plane profile $\nabla J_p^i(x) = R^i = [1,\ 2,\ 3]^\top$ for all i.

A. No-Noise Case

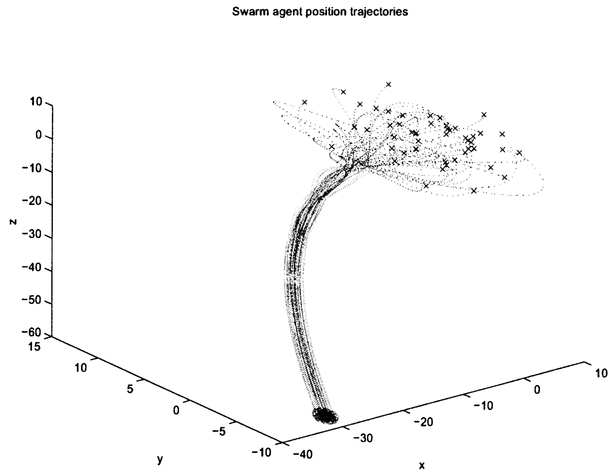
All the simulations in this case are run for 20 s. The position and velocity trajectories of the swarm agents with the normal parameters are shown in Fig. 2. All the agents are assigned initial velocities and positions randomly. At the beginning of the simulation, they appear to move around erratically. But soon, they swarm together and continuously reorient themselves as a group to slide down the plane profile. Note how these agents gradually catch up with each other while still keeping mutual spacing. Recall from SectionIII that for this case $v^i(t) \to -k_f/kR$ for all i as $t \to \infty$, and this can be seen from Fig. 2(b) since the final velocity of each swarm agent is indeed $-[1,\ 2,\ 3]^\top$.

Next, we change the value of some of the parameters to show their impact on the system behavior.

Fig. 3(a) and (b) show the results of keeping all normal parameters unchanged except for increases of k_r to 1000 and r_s to 1, respectively. Since both k_r and r_s are parameters affecting the repulsion range of each agent, as expected we find that the final swarm size becomes larger than in the previous case, while the swarm velocity and settling speed do not change much. We also investigated the number of collisions occurring between the agents. As expected, when k_r and r_s increase, the number of collisions decreases. Effects of other parameters are also as expected.

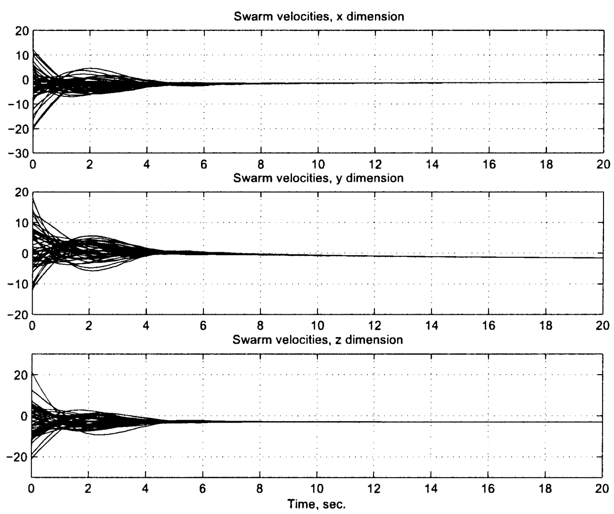
B. Noise Case

Now, we will consider the case when noise exists. In our simulations the solutions of Duffing's Equation are used as "noise" so that the noise is guaranteed to be differentiable. Of course many other choices are possible, e.g., ones that lead to errors on



(a) Agent position trajectories.

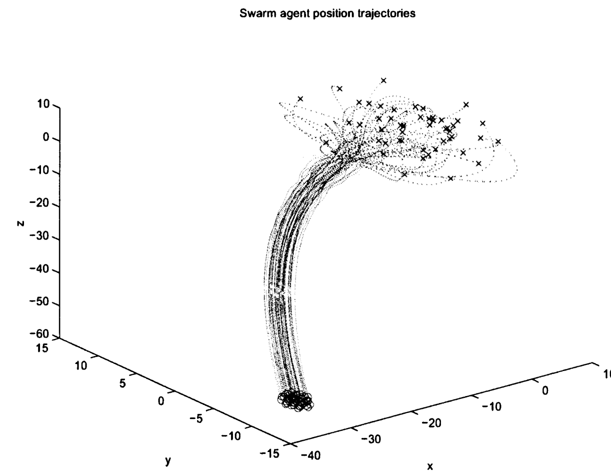
(a)



(b) Agent velocity trajectories.

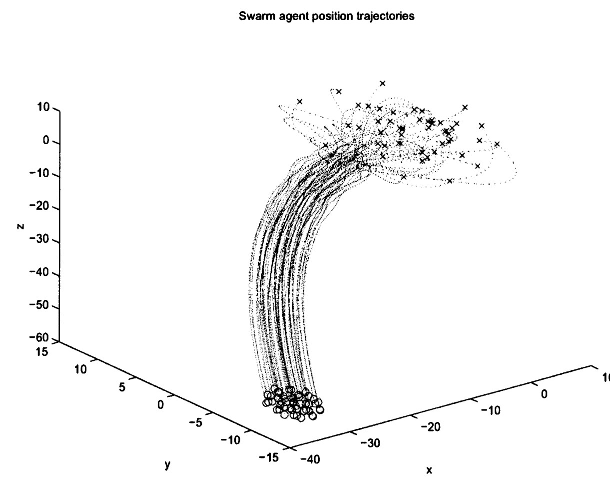
(b)

Fig. 2. No noise case with normal parameters. (a) Agent position trajectories. (b) Agent velocity trajectories.



(a) Agent position trajectories ($k_r = 1000$, $r_s = 0.1$).

(a)



(b) Agent position trajectories ($r_s = 1$, $k_r = 10$).

(b)

Fig. 3. No noise case with parameters changed. (a) Agent position trajectories $(K_r = 1000, r_s = 0.1)$. (b) Agent velocity trajectories $(r_s = 1, K_r = 10)$.

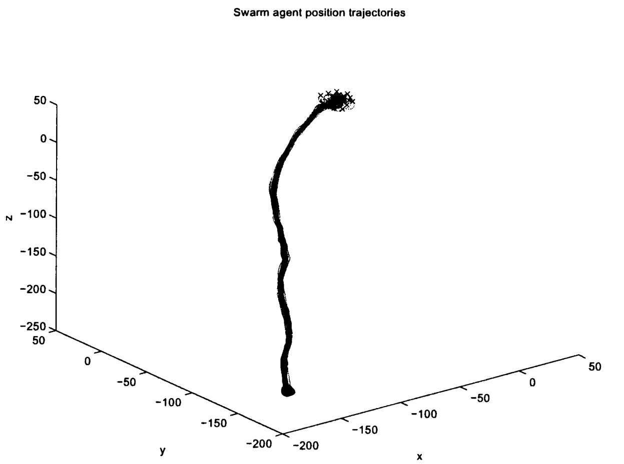
a higher or lower frequency spectrum. Duffing's equation is in the form

$$\ddot{\vartheta} + \delta_D \dot{\vartheta} - \vartheta + \vartheta^3 = \gamma_D \cos(\omega_D t).$$

In the simulations we use $\delta_D = 0.25$, $\gamma_D = 0.30$ and $\omega_D = 1.0$ so that the solution ϑ of Duffing's Equation demonstrates chaotic behavior. We will simulate many such equations to generate noise on position, velocity, and resource profile gradient sensing. We denote by ϑ_i the solution to the ith Duffing's Equation that we simulate. Note that the magnitude of ϑ is always bounded by a value of 1.5. Thus, we can easily change the noise bounds with some scaling factors. For example, in the case of noise with a linear bound, the position sensing noise is generated by $d_p^i = D_{p_1} \vartheta_1^i / 1.5 \|E^i\| + D_{p_2} \vartheta_2^i / 1.5$ so that (12) is satisfied. In this case, we run the simulations for 80 s. All the normal parameters used in the no-noise case are kept unchanged except the number of agents in the swarm in certain simulations, which is specified in the relevant figures. Figs. 4 and 5 illustrate the case with linear noise bounds for a typical simulation run. The noise bounds are $D_{p_1} = D_{v_1} = 0.05$, $D_{p_2} = D_{v_2} = 1$, and $D_f = 10$, respectively. According to the Grunbaum principle [13], [14], forming a swarm may help the agents go down the gradient of the nutrient profile without being significantly distracted by noise. Fig. 4 shows that the existence of noise does affect the swarm's ability to follow the profile, which is indicated by the oscillation of the position and velocity trajectories. However, with all the agents working together, especially when the agents number N is large, they are able to move in the right direction and thus, minimize the negative effects of noise. In comparison, Fig. 5 shows the case when there is only one agent. Since the single agent cannot benefit from the averaging effects possible when there are many agents, the noise more adversely affects its performance in terms of accurately following the nutrient profile.

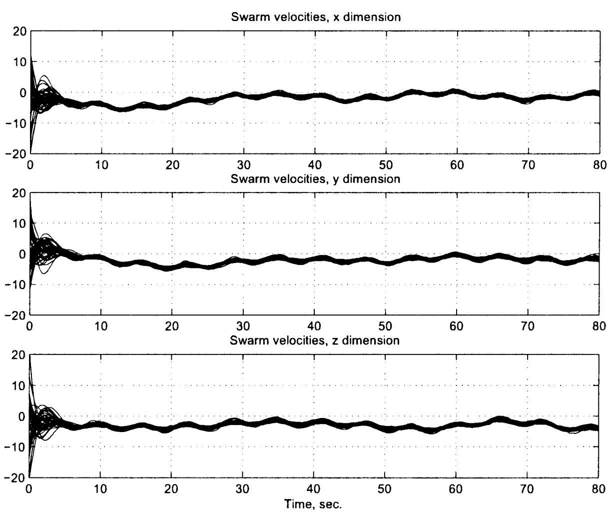
VI. Concluding Remarks

We have derived conditions under which social foraging swarms maintain cohesiveness and follow a resource profile even in the presence of sensor errors and noise on the profile.



(a) Agent position trajectories.

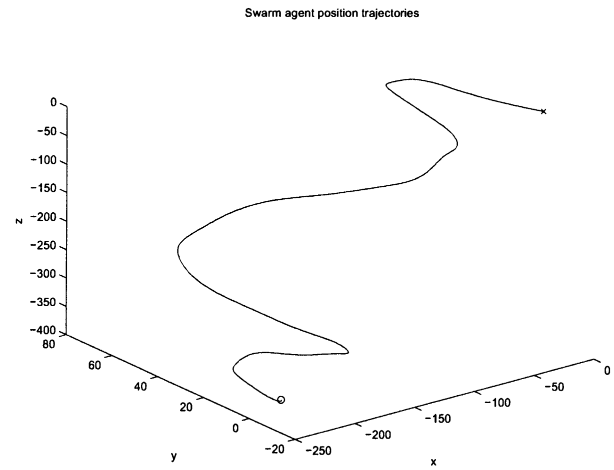
(a)



(b) Agent velocity trajectories.

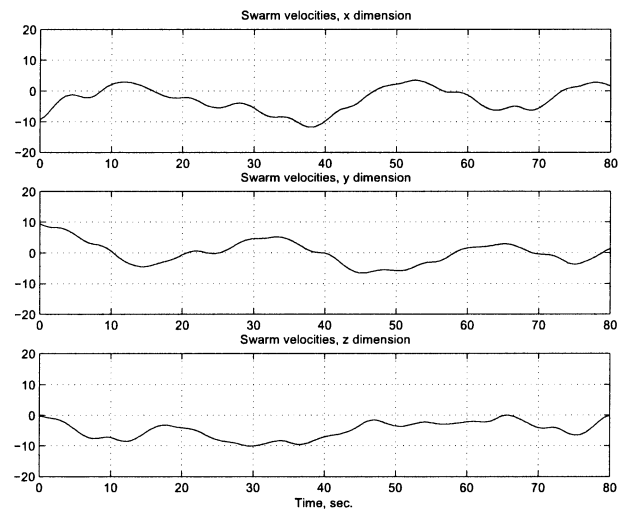
(b)

Fig. 4. Linear noise bounds case $(N = 50)$. (a) Agent position trajectories. (b) Agent velocity trajectories.



(a) Agent position trajectories.

(a)



(b) Agent velocity trajectories.

(b)

Fig. 5. Linear noise bounds case $(N = 1)$. (a) Agent position trajectories. (b) Agent velocity trajectories.

We also studied special types of noise and the case where all agents are identical. Although we only studied one type of attraction and repulsion function, the results can be extended to other classes using approaches such as the ones in [12] and [16]. Moreover, while we only studied the plane profile, extensions to profiles with other shapes such as those studied in [17]are possible. In fact any profile that is smooth and has a known finite slope can be fit into the framework without changing our major results except part of Theorem 3. Specifically, the conclusion that the velocities of all agents approach a constant [prescribed by (44)] may not hold any more.

Our simulations illustrated advantages of social foraging in large groups relative to foraging alone since they show that a noisy resource profile can be more accurately tracked by a swarm than an individual [13], [14]. Moreover, the simulations produced agent trajectories that are curiously reminiscent of those seen in biology (e.g., by some insects). It would be interesting to determine if the model here, with appropriately chosen parameters, is an acceptably accurate representation for some social organizms and whether the predictions of the analysis would also accurately represent their group-level behavior.

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