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An Optimal Volume Ellipsoid Algorithm for Parameter Set Estimation

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Abstract—In this note, a recursive ellipsoid algorithm is derived for parameter set estimation of a SISO linear time-invariant system with bounded noise. The algorithm objective is in seeking the minimal volume ellipsoid bounding the feasible parameter set. Cast in a recursive framework, where a minimal volume ellipsoid results at each recursion, the algorithm extends a result due to Khachian in 1979 in which a technique was developed to solve a class of linear programming problems. This extension and application to the parameter set estimation problem has intuitive geometric appeal and is easy to implement. Comparisons are made to the Optimal Bounding Ellipsoid (OBE) algorithm of Fogel and Huang, and the results are demonstrated via computer simulations.

I. INTRODUCTION

The concept of parameter set estimation in system identification has evolved over the past two decades. The motive in parameter set estimation is to identify a feasible set of parameters which is consistent with the measurement data and the model structure used. One can interpret the set estimate as some nominal parameter estimate accompanied by a quantification of the uncertainty, either parametrically or nonparametrically, around the nominal model. An important feature in the parameter set estimation is the guaranteed inclusion of the true plant which is not exactly known.

In this note, an optimal recursive ellipsoid algorithm for parameter set estimation is developed. This result is based on the Khachian ellipsoid algorithm [1] developed for solving the linear programming problem. At each recursion, the smallest volume ellipsoid bounding a convex polytope defined by the bounded noise is found. This result is distinct from the innovative OBE algorithm due to Fogel and Huang [2] in that every new ellipsoid in the course of updating is optimal under no constraints. This is in contrast with the OBE algorithm in which the optimization is subjected to the constraint that the center of the ellipsoid is a "modified" recursive least squares estimate.

Though both the present algorithm and the OBE algorithm have similar recursive equations for implementation, the optimal bounding algorithm of this note gives an appealing geometrical interpretation of the ellipsoid bounding the convex set of interest and ultimately results in the true minimum volume bounding ellipsoids.

The main contribution of this note is the extension of the single hyperplane cut in Khachian's ellipsoid algorithm to the parameter set estimation problem where the feasible set of estimates are constrained between two parallel hyperplanes. Khachian's ellipsoid algorithm handles multiple constraints sequentially whereas here, multiple constraints are handled pairwise. In fact, our algorithm can be reduced to Khachian's ellipsoid algorithm. The algorithm derived in this note offers 1) careful and correct development for the parameter set estimation problem, with example and convergence result; 2) an entirely different proof, offering more geometrical insight into the problem, than the algorithms which have appeared in [3], [4] for solving linear programming problems. Moreover, the similarities between our algorithm and the OBE algorithm in [2] are noted, and a qualitative comparison is included.

II. PROBLEM STATEMENT

Consider a SISO ARX model

$$y_k = -\sum_{i=1}^{n} a_i y_{k-i} + \sum_{j=0}^{m} b_j u_{k-j} + v_k \qquad (1)$$

$$= \theta^T \phi_k + v_k \qquad (2)$$

where $\theta^T = [a_1, \cdots, a_n, b_0, \cdots, b_m]$ is the parameter vector to be estimated; $\phi_k = [-y_{k-1}, \cdots, -y_{k-n} u_k, \cdots, u_{k-m}]^T$ is the regression vector containing the past inputs, $u(\cdot)$, and outputs, $y(\cdot)$; n and m are the number of system poles and zeros, respectively; v_k is a sequence of bounded disturbances/noise corrupting the system output with $|v_k| \leq \gamma$ for all $k \geq 0$. It is assumed that n, m, and γ are known *a priori*.

Let $\mathcal{F} \in \Re^{n+m+1}$ be a set such that all $\theta \in \mathcal{F}$ are feasible parameter estimates of the plant which are consistent with the measurements. That is

$$\mathcal{F} = \{\theta: |y_k - \theta^T \phi_k| \leq \gamma, k = 0, \cdots, N\}. \qquad (3)$$

The problem of parameter set estimation is to find $\mathcal{F}$ explicitly in the parameter space. In general, $\mathcal{F}$ is an irregular convex set, so we wish to find a more manageable convex set to over-bound $\mathcal{F}$ for the purpose of system analysis and control. Ellipsoids are commonly used to bound $\mathcal{F}$ for their simplicity in mathematical representation and manipulation in computation. It is therefore desired to find the smallest ellipsoid to contain the set $\mathcal{F}$ where the hyper-volume of an ellipsoid is used to measure "smallness."

III. THE OVE ALGORITHM

In this section, Khachian's ellipsoid algorithm is extended to the problem of parameter set estimation. For convenience, we will refer to the new algorithm as the Optimal Volume Ellipsoid (OVE) algorithm. For the set of inequality constraints in $\mathcal{F}$ defined in the last section, consider a pair of constraints

$$|y_{k+1} - \theta^T \phi_{k+1}| \leq \gamma \qquad (4)$$

and let the set $\mathcal{F}_{k+1}$ be defined as follows:

$$\mathcal{F}_{k+1} = \{\theta: |y_{k+1} - \theta^T \phi_{k+1}| \leq \gamma\}. \qquad (5)$$

Geometrically, $\mathscr{F}_{k+1}$ is the region between the two parallel hyperplanes defined in (4). The set estimation problem is then stated as: Given an ellipsoid E_k, find another ellipsoid E_{k+1} with minimal volume, such that E_{k+1} contains $E_k \cap \mathscr{F}_{k+1}$, for $k = 0, \cdots, N$, where N is the number of data records. Mathematically, the optimization problem becomes

$$\min \{ \mathrm{vol}\, (E_{k+1}) \colon E_{k+1} \supset E_k \cap \mathscr{F}_{k+1} \}.$$

Define E_k and E_{k+1} as

$$E_k = \left\{ \theta \colon (\theta - \theta_k)^T P_k^{-1} (\theta - \theta_k) \le 1; \quad \theta \in \Re^r \right\} \quad (6)$$

and

$$E_{k+1} = \left\{ \theta \colon (\theta - \theta_{k+1})^T P_{k+1}^{-1} (\theta - \theta_{k+1}) \le 1; \quad \theta \in \Re^r \right\} \quad (7)$$

where $r = n + m + 1$ and θ_k is the center estimate of the ellipsoid at time k.

In the derivation of the OVE algorithm, an affine transformation [1],

$$\theta = \theta_k + J\hat{\theta} \quad (8)$$

is used to simplify the analysis where $\theta \in \Re^r$ is any vector in the parameter space, $\hat{\theta}$ is the parameter vector in the affine transformed coordinate and $P_k = JJ^T$. Through this transformation, the ellipsoid E_k is mapped to the unit radius hypersphere centered at the origin. The set estimation problem for a specific value of k reduces to finding the minimal volume ellipsoid containing the intersection between a unit radius hypersphere and two parallel hyperplanes defined by $\hat{\mathscr{X}}$, the affine transformation of $\mathscr{F}_{k+1}$. Let

$$\mathscr{S} = \{ \hat{\theta} \colon \hat{\theta}^T \hat{\theta} \le 1 \} \quad (9)$$

and

$$\hat{\mathscr{X}} = \left\{ \hat{\theta} \colon \frac{\hat{\phi}^T \hat{\theta}}{(\hat{\phi}^T \hat{\phi})^{1/2}} \le \alpha \quad \text{and} \quad \frac{\hat{\phi}^T \hat{\theta}}{(\hat{\phi}^T \hat{\phi})^{1/2}} \ge \alpha - 2\beta \right\} \quad (10)$$

where $\hat{\phi}$ is the transformed ϕ_{k+1}, and α, β are parameters defining the location of the two parallel hyperplanes defined in $\hat{\mathscr{X}}$ which are 2β ($\beta > 0$) apart (we will define these parameters later as related to the original parameter set estimation problem). In what follows, we wish to find an ellipsoid

$$\hat{E} = \left\{ \hat{\theta} \colon \left(\hat{\theta} - \hat{\theta}_0 \right)^T \hat{A}^{-1} \left(\hat{\theta} - \hat{\theta}_0 \right) \le 1 \right\} \quad (11)$$

such that $\mathscr{S} \cap \hat{\mathscr{X}} \subset \hat{E}$ and the volume of $\hat{E}$, $\mathrm{vol}(\hat{E})$, is minimized.

For the purpose of analysis, define

$$\hat{H}_1 = \left\{ \hat{\theta} \colon \frac{\hat{\phi}^T \hat{\theta}}{(\hat{\phi}^T \hat{\phi})^{1/2}} \le \alpha \right\}$$

$$\hat{H}_2 = \left\{ \hat{\theta} \colon \frac{\hat{\phi}^T \hat{\theta}}{(\hat{\phi}^T \hat{\phi})^{1/2}} \ge \alpha - 2\beta \right\}.$$

If we denote the superscript * as the boundary, then from these definitions, $\hat{\mathscr{X}}$ is the region between the two hyperplanes, $\hat{H}_1^*$ and $\hat{H}_2^*$ are the two hyperplanes 2β apart with $\hat{H}_1^*$ at a distance α from the origin of $\mathscr{S}$, and the vector $\hat{\phi}$ is orthogonal to both the hyperplanes. Fig. 1 depicts the scenario. Before we proceed, we need the following lemma and theorem.

Lemma 1: Given that $\mathscr{S} \cap \hat{\mathscr{X}}$ is symmetrical about the $\hat{\theta}_1$ axis (one of the coordinate axes in $\hat{\theta}$ system), achieved by adding one more rotation to the affine transformation to align $\hat{\phi}$ with the
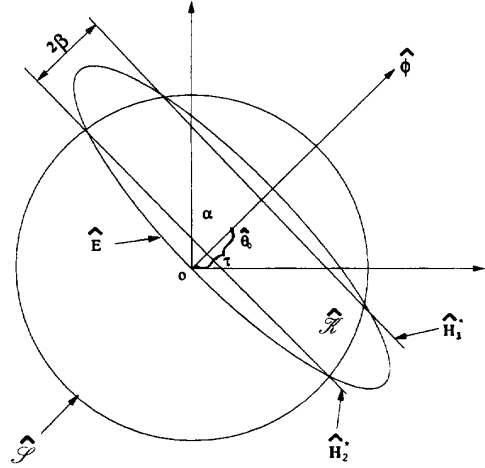

Fig. 1. Ellipsoid bounding the intersection of a hypersphere and two parallel hyperplanes.

coordinate axis $\hat{\theta}_1$, the minimal volume ellipsoid containing $\mathscr{S} \cap \hat{\mathscr{X}}$ must also be symmetrical about the $\hat{\theta}_1$ axis.

Theorem 1: Given $\mathscr{S}$ and $\hat{\mathscr{X}}$, the minimum volume ellipsoid $\hat{E}$ bounding $\mathscr{S} \cap \hat{\mathscr{X}}$ must satisfy the following conditions:

$$\hat{H}_1^* \cap \mathscr{S} = \hat{H}_1^* \cap \hat{E} \quad \text{and} \quad \hat{H}_2^* \cap \mathscr{S} = \hat{H}_2^* \cap \hat{E}. \quad (12)$$

Essentially, these conditions imply that $\hat{E}^*$, the surface of $\hat{E}$, must pass through the intersecting points between $\mathscr{S}^*$ (the surface of $\mathscr{S}$) and the two parallel hyper-planes, $\hat{H}_1^*$ and $\hat{H}_2^*$. Qualitatively, since $\mathscr{S} \cap \hat{\mathscr{X}}$ is symmetrical about $\hat{\phi}$, the smallest volume ellipsoid must also be symmetrical about $\hat{\phi}$, and given any ellipsoid symmetrical about $\hat{\phi}$ and containing $\mathscr{S} \cap \hat{\mathscr{X}}$ but not satisfying (12), a smaller ellipsoid can be constructed such that it not only contains $\mathscr{S} \cap \hat{\mathscr{X}}$, but also satisfies (12). For the proof of Lemma 1 and Theorem 1 (as well as for subsequent proofs), the reader is referred to [5]. We now state the main result for the OVE algorithm.

Theorem 2: For the sets $\mathscr{S}$ and $\hat{\mathscr{X}}$, if $|\alpha| \le 1$ and $|2\beta - \alpha| \le 1$, then the following parameters will result in a minimal volume $\hat{E}$ that contains $\mathscr{S} \cap \hat{\mathscr{X}}$

$$\hat{\theta}_0 = \tau \hat{\phi} / (\hat{\phi}^T \hat{\phi})^{1/2} \quad (13)$$

$$\hat{A} = \delta \left(I - \frac{\sigma \hat{\phi} \hat{\phi}^T}{\hat{\phi}^T \hat{\phi}} \right) \quad (14)$$

provided $\sigma \ge 0$ where
i) if $\alpha \ne \beta$:

$$\delta = \frac{(\tau + 1)^2 (\beta - \alpha) - \tau(1 + \alpha)(2\beta - \alpha - 1)}{\tau + \beta - \alpha} \quad (15)$$

$$\sigma = \frac{-\tau}{\beta - \alpha} \quad (16)$$

and τ is the real solution of

$$(r + 1)\tau^2 + \left\{ \frac{(1 + \alpha)(\alpha - 2\beta + 1)}{\beta - \alpha} + 2[r(\beta - \alpha) + 1] \right\} \tau + r\alpha(\alpha - 2\beta) + 1 = 0 \quad (17)$$

such that $\alpha - 2\beta < \tau < \alpha$;

ii) if $\alpha = \beta$:

$$\delta = \frac{r}{r-1}(1 - \beta^2) \tag{18}$$

$$\sigma = \frac{1 - r\beta^2}{1 - \beta^2} \tag{19}$$

$$\tau = 0. \tag{20}$$

In the above, note that I is an r by r identity matrix and $r = n + m + 1$; moreover, if $\sigma < 0$, the minimal volume $\hat{E}$ is $\mathscr{S}$ itself.

Proof: The proof is divided into two parts: i) the parameters required for a minimal volume $\hat{E}$ to contain $\hat{H}_1^* \cap \mathscr{S}$ and $\hat{H}_2^* \cap \mathscr{S}$, ii) the condition for $\hat{E}$ to actually contain $\mathscr{S} \cap \mathscr{H}$. Details of the proof appear in [5].

Remarks:

1) When $|\alpha| \leq 1$, it means that $\hat{H}_1^*$ must cut, or at least touch, the hyper-sphere $\mathscr{S}$, whereas $|2\beta - \alpha| \leq 1$ means that $\hat{H}_2^*$ must cut, or at least touch the hyper-sphere $\mathscr{S}$. The case $\alpha > 1$ or $2\beta - \alpha > 1$ means that $\hat{H}_1^*$ or $\hat{H}_2^*$ does not cut $\mathscr{S}$, respectively. In the first case, β can be reset to $\beta - (\alpha - 1)/2$ and then reset α to 1 if $\alpha > 1$ to make up a new hyper-plane parallel to $\hat{H}_1^*$ but touching $\mathscr{S}$. In the second case, β is set to $(1 + \alpha)/2$ if $2\beta - \alpha > 1$ to make up a hyper-plane parallel to $\hat{H}_2^*$ but touching $\mathscr{S}$.

2) If $2\beta - \alpha = 1$, then $\tau = 1 - (2r\beta/r + 1)$, and if $\alpha = 1$, then $\tau = 2r\beta/(r + 1) - 1$ for a minimal volume ellipsoid. These special cases are equivalent to those using the Khachian algorithm in [6].

Theorem 3 (The OVE Algorithm): For the system in (2) with the bounded noise constraint (3), the equations for θ_{k+1} and P_{k+1} that result in the minimal volume ellipsoid E_{k+1} bounding the intersection between the given $\mathscr{F}_{k+1}$ and E_k defined in (5) and (6), respectively, are as follows:

$$\theta_{k+1} = \theta_k + \frac{\tau_k P_k \phi_{k+1}}{(\phi_{k+1}^T P_k \phi_{k+1})^{1/2}}$$

$$P_{k+1} = \delta_k \left(P_k - \sigma_k \frac{P_k \phi_{k+1} \phi_{k+1}^T P_k}{\phi_{k+1}^T P_k \phi_{k+1}} \right)$$

where τ_k, δ_k, σ_k are, respectively, equivalent to τ, δ and σ in (15)–(17) or (18)–(20) depending on the values of α and β in Theorem 2. These values of α and β are given by

$$\beta = \frac{\gamma}{\sqrt{\phi_{k+1}^T P_k \phi_{k+1}}} \quad \text{and} \quad \alpha = \frac{y_{k+1} + \gamma - \phi_{k+1}^T \theta_k}{\sqrt{\phi_{k+1}^T P_k \phi_{k+1}}}.$$

If $\alpha > 1$, reset β to $\beta - (\alpha - 1)/2$ and then reset α to 1; on the other hand, if $2\beta - \alpha > 1$, reset β to $(1 + \alpha)/2$. The algorithm can be initiated with a sufficiently large E_0 containing the feasible parameter set, where $\theta_0 = 0$ and $P_0 = (1/\epsilon)I$ (with $0 < \epsilon \ll 1$) are typical starting values.

Proof: Consider the affine transformation discussed earlier and let $P_{k+1} = J\hat{A}J^T$ and $\phi_{k+1} = J^{-T}\hat{\phi}$, then E_k is mapped to $\mathscr{S}$ and E_{k+1} is mapped to $\hat{E}$. Through some algebraic manipulations, the result follows [5].

IV. OVE CONVERGENCE

Theorem 4: Consider the OVE Algorithm and define the Lyapunov function

$$V_k \equiv \det P_k$$

and

$$\tilde{\theta}_k \equiv \theta^* - \theta_k$$

where θ^* is the true parameter vector. Then

i) $V_{k+1} \leq V_k$

ii) $|e_k| < \gamma$ as $k \to \infty$

iii) $\|\hat{\theta}_k\|_2^2 \leq \bar{\sigma}(P_k)$ where $\bar{\sigma}(P_k)$ denotes the largest singular value of P_k.

Further, if there exists an integer N_1 ($\geq r$, the number of unknown parameters), and positive real numbers c_1 and c_2 such that

$$c_1 I \leq \sum_{i=k+1}^{k+N_1} \phi_i \phi_i^T \leq c_2 I$$

for all k, then

$$\|\theta^* - \theta_k\|_2^2 \leq \frac{4\gamma^2 N_1}{c_1}$$

as $k \to \infty$.

Proof: Details appear in [5] and are omitted here.

V. COMPARISONS TO OBE

The OBE algorithm in [2] is based on the result in [7] to find an ellipsoid to contain the intersection of another two ellipsoids, and a modified RLS type update of the center of ellipsoids is adopted. The derivation of the OVE algorithm takes on a different approach based on a geometrical point of view. The similarities in the form of the expression for P_{k+1} indicate that E_{k+1} in both the OVE and OBE algorithms have the same orientation with one of the axes parallel to $\hat{\phi}$ in the affine transformed coordinate, given the same E_k. From the geometrical viewpoint, if E_k is mapped to a unit radius hypersphere $\mathscr{S}$ and E_{k+1} to $\hat{E}$ through an affine transformation, it is seen that the center of ellipsoid $\hat{E}$ in the OBE algorithm does not necessarily lie on the vector $\hat{\phi}$ because of the dependence of θ_{k+1} on P_{k+1}, whereas in the OVE algorithm the center of $\hat{E}$ always lies on $\hat{\phi}$. The main difference, and this is of paramount importance, is the location of the center of the ellipsoid. The extra RLS type constraint that is imposed on the center estimate of the ellipsoids in the OBE algorithm essentially precludes the satisfaction of the necessary condition for a minimal volume ellipsoid E_{k+1} to contain $E_k \cap \mathscr{F}_{k+1}$, where $E_k \cap \mathscr{F}_{k+1}^* = E_{k+1} \cap \mathscr{F}_{k+1}^*$ in which $\mathscr{F}_{k+1}^*$ is the boundary of $\mathscr{F}_{k+1}$.

The OVE algorithm also applies to the case when one of the hyperplanes does not cut the recursive ellipsoid which may be crucial when the goal is to find the smallest set; this is also studied for the Modified OBE (MOBE) algorithm in [8]. Essentially, OBE and MOBE are equivalent except when one of the hyperplanes does not cut the recursive ellipsoid. Both the OVE and OBE algorithms update estimates selectively according to the received data.

As a final note on comparing the OBE and the OVE algorithms, the numbers of multiplication and addition operations required in the information evaluation are on the order of r^2 for both algorithms where r is the number of unknown system parameters. However, for updating estimates the OBE algorithm requires $6r^2$ multiplications whereas in the OVE algorithm, only $5r^2$ multiplications are required (and no more additions are required for the OVE algorithm). The extra computation for the

OBE algorithm is basically due to the need to compute P_{k+1}. The performance of both algorithms is compared by way of examples in the next section.

VI. SIMULATION RESULTS

Consider a second-order system which represents a flexible structure truss model containing only the first x-bending mode [9] with the following discretized transfer function

$$Y(z) = \frac{0.1156(z - 1)}{z^2 - 1.55z + 0.8267} U(z)$$

$$+ \frac{1}{z^2 - 1.55z + 0.8267} V(z). \quad (21)$$

In the simulation, $|v_k| \le 0.05$ and the S/N is 20 dB, $N = 100$ data points are taken. To compare the OVE and OBE algorithms, the same input sequence and noise sequence are used in the two simulations. The following notation is adopted: θ_{true} is the true estimate, θ_k^{OBE} is the center estimate of the ellipsoid associated with the OBE algorithm and θ_k^{OVE} is the center estimate of the ellipsoid associated with the OVE algorithm at time k; the parameter interval associated with the ellipsoids are denoted as I^{OBE} and I^{OVE} accordingly for the final ellipsoids. The results of the two algorithms are as follows

$$\theta_{\text{true}} = \begin{bmatrix} -1.55 \\ 0.8267 \\ 0.1156 \\ -0.1156 \end{bmatrix}, \quad (22)$$

$$\theta_{\text{N}}^{\text{OBE}} = \begin{bmatrix} -1.538 \\ 0.8226 \\ 0.1155 \\ -0.1158 \end{bmatrix}, \quad \theta_{\text{N}}^{\text{OVE}} = \begin{bmatrix} -1.5455 \\ 0.8244 \\ 0.1157 \\ -0.1182 \end{bmatrix} \quad (23)$$

and the parameter intervals are

$$I^{\text{OBE}} = \begin{bmatrix} -1.72 & \leftrightarrow & -1.356 \\ 0.6702 & \leftrightarrow & 0.9751 \\ 0.1004 & \leftrightarrow & 0.1311 \\ -0.1405 & \leftrightarrow & -0.0905 \end{bmatrix}, \quad (24)$$

$$I^{\text{OVE}} = \begin{bmatrix} -1.6596 & \leftrightarrow & -1.4341 \\ 0.7223 & \leftrightarrow & 0.9266 \\ 0.1062 & \leftrightarrow & 0.1253 \\ -0.1347 & \leftrightarrow & -0.1016 \end{bmatrix}. \quad (25)$$

Note that both of the intervals contain the true parameters. Fig. 2 shows the volume of ellipsoids, E_k^{OVE}, E_k^{OBE} and E_k^{MOBE} due to the OVE algorithm and the OBE algorithm, respectively, over 100 data points. It is clear from the figure that ellipsoids from the OVE algorithm are always smaller than that from the OBE or MOBE algorithm (they are always less than half of the volume of the OBE ellipsoids). The MOBE algorithm improves only slightly over the OBE algorithm. However, all these algorithms guarantee monotonic nonincreasing volume of the ellipsoids, consistent with the theory.

Moreover, from the viewpoint of the parameter interval, by putting an orthotope which is orthogonal to the parameter axes and tightly overbounding the ellipsoid, Fig. 3 shows the interval for each parameter at each iteration for the two algorithms. In the plot, the upper and lower intervals of each parameter are indicated for each algorithm. It can be seen that the OVE algorithm almost always gives tighter parameter bounds than does the OBE algorithm. Exceptions to this occur when the size
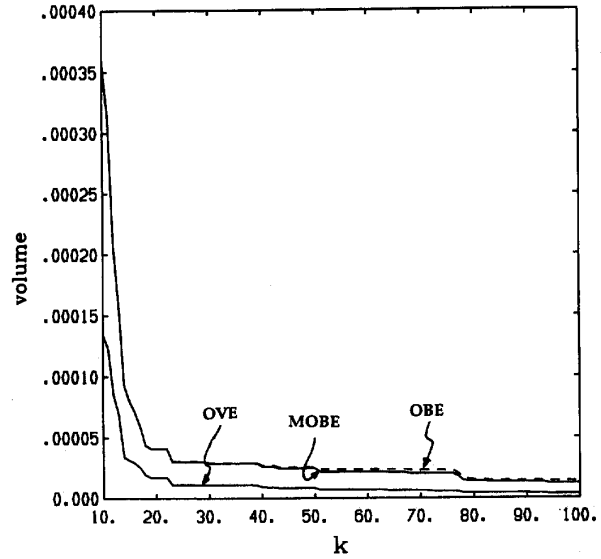

Fig. 2. Volume of E_k^{OVE}, E_k^{OBE}, and E_k^{MOBE}.
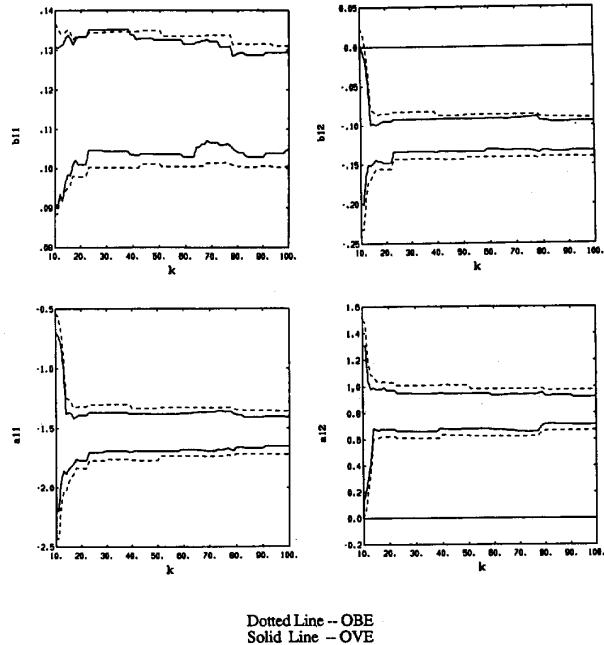

Dotted Line -- OBE
Solid Line -- OVE

Fig. 3. Parameter intervals for both algorithms.

of some of the axes of an ellipsoid are reduced, while other axes are expanded to contain a certain convex set, possibly causing a larger bound in some of the parameters at early iterations. Eventually, tighter bounds are noted in the recursion as in Fig. 3 and (25).

VII. CONCLUSION

In this note, the Khachian ellipsoid algorithm for the linear programming problem is extended to the problem of parameter set estimation with bounded noise. We first showed that a

minimal volume ellipsoid bounding the intersection between a hypersphere and two parallel hyperplanes must tightly contain the intersecting points between the surface of the hypersphere and the hyperplanes. Using this result, we derived the minimal volume ellipsoid containing the intersection between a hypersphere and two parallel hyperplanes, resulting in the new recursive algorithm, OVE, for parameter set estimation. Convergence results of the OVE algorithm are also given. It is noted that the OVE algorithm has similar form to the well known OBE algorithm. The OVE algorithm possesses several attractive features in comparison to the OBE algorithm: 1) the OVE algorithm is rich in geometrical interpretations; 2) the formulation of OVE algorithm is flexible enough to accommodate the case when one of the hyperplanes defined in $\mathscr{F}_{k+1}$ does not cut the ellipsoid E_k; 3) the OVE algorithm requires no additional computational complexity than the OBE algorithm; 4) The OVE algorithm results in the smallest volume ellipsoid E_{k+1} bounding $E_k \cap \mathscr{F}_{k+1}$ without any constraints. Several extensions to the OVE algorithm for parameter set estimation and control are under investigation, including an application to interconnected systems [10], time-varying noise bounds, input synthesis, and ARMAX models with measurement noise [11].

With regard to other similar results which have recently appeared, the OVE algorithm offers correct development for the parameter set estimation problem in a geometrical approach with example and convergence result, as distinct from the results in [3], [4]. Moreover, the note by Pronzato, *et al.* [12], alludes to the EPC (parallel cut) algorithm for bounded error estimation, which is in fact equivalent to the OVE algorithm (as opposed to the OBE algorithm), in that both were derived from the Khachian algorithm. However, we have developed the OVE algorithm entirely from a geometric point of view, with convergence results, and offer the novel application to the parameter set estimation problem in this note.

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Optimal Use of an Extra Server in a Two Station Tandem Queueing Network

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Abstract—Consider a two station tandem queueing system, with given numbers of customers initially at each station and no arrivals. There is a fixed server at each station, but also an additional server that can be dynamically allocated to wherever its use will do most good. There are differing linear holding costs at each station, and the aim is to use the extra server to minimize the expected total holding cost incurred until the system empties. We show that if either the extra server may be switched between the two stations at any time, or if it is restricted in use to just one station, where it may be turned on or off, then the optimal use of the server is such that after a service completion at one station, the effort devoted there never increases, and the effort devoted to the other station never decreases.

I. INTRODUCTION

Consider a system of two queues in tandem, with no arrivals after time zero. Linear holding costs, of c_1 and c_2 per customer, are incurred at the first and second stations, respectively. There is a fixed server at each of the stations, and an additional server that may either be switched off, or whose effort may be dynamically allocated between the stations. The objective is to minimize the expected total holding cost until the system empties, subject possibly to the constraint that the extra server may not be used at one of the stations. This is an example of what have been called *clearing systems*. Service times of customers served by the two fixed servers and the additional server are exponentially distributed with parameters μ_1, μ_2 and μ, respectively. All service times are independent. We assume that movement of the additional server may take place instantaneously and without penalty, and that no cost is imposed for use of the servers. Thus, we have three possible cases to consider: a) the additional server is constrained so that it may be used only at the upstream queue or not at all; b) the server is constrained to operate only at the downstream queue; and c) the allocation of the server is unrestricted. In Fig. 1, these correspond respectively to the constraints $a_2 = 1$, $a_1 = 1$ and $a_1 + a_2 = 1$. We see that case b) is trivial, as the additional server will never be idle while it is possible to reduce the instantaneous holding cost, and in case c) it is also never optimal for the extra server to be idle, so the decision problem is simply whether it should be working at station 1 or 2.

Control of a two station tandem queueing system was first discussed by Rosberg, Varaiya and Walrand [5]. In the context of