Stability Analysis of One-Dimensional Asynchronous Swarms

Yang Liu, Kevin M. Passino, and Marios Polycarpou

***Abstract*—Coordinated swarm behavior in certain types of animals can also occur in groups of autonomous vehicles. Swarm "cohesiveness" is characterized as a stability property. Conditions for one-dimensional asynchronous swarms to achieve collision-free convergence even in the presence of sensing delays and asynchronism during movements are provided. Each finite-size swarm member has proximity sensors and neighbor position sensors that only provide delayed position information. Such stability analysis is of fundamental importance if one wants to understand the coordination mechanisms for "platoons" of autonomous vehicles, where intermember communication channels are less than perfect and collisions must be avoided.**

***Index Terms*—Asynchronism, communication delay, discrete-event systems, stability, swarms.**

I. Introduction

A variety of organizms have the ability to cooperatively forage for food while trying to avoid predators and other risks. For instance, when a school of fish searches for prey, or if it encounters a predator, the fish often make coordinated maneuvers as if the entire group were one organizm [1]. Analogous behavior is seen in flocks of birds, herds of wildebeests, groups of ants, and swarms of social bacteria [2], [3]. We call this kind of aggregate motion "swarm behavior." A high-level view of a swarm suggests that the organizms are cooperating to achieve some purposeful behavior and achieve some goal. Naturalists and biologists have studied such swarm behavior for decades.

Recently, there has been a growing interest in biomimicry of the mechanisms of foraging and swarming for use in engineering applications since the resulting swarm intelligence can be applied in optimization [2], [3], robotics [4], [5], traffic patterns in intelligent transportation systems [6], [7], and military applications [8]. For instance, there has been a growing interest in groups (swarms) of flying vehicles [9]. It has been argued that a swarm of robots can accomplish some tasks that would be impossible for a single robot to achieve. Particular research includes that of [5], who introduced the concept of cellular robotic systems, and the related study in [10]. The behavior-based control strategy put forward by Brooks [11] is quite well known and it has been applied to collections of simple independent robots. Mataric [12] describes experiments with a homogeneous population of robots acting under different communication constraints. Suzuki and Yamashita [13] considered a number of two-dimensional problems of formation of geometric patterns with distributed anonymous mobile swarm robots.

In this note, we are interested in mathematical modeling and analysis of stability properties of swarms. We think of stability as characterizing the cohesiveness of the swarm as it moves. Stability is a basic qualitative property of swarms since if it is not present, then it is maybe impossible for the swarm to achieve any other group objective. Stability analysis of swarms is still an open problem but there have been several areas of relevant progress. Jin *et al.* in [14] studied stability of synchronized distributed control of one-dimensional (1-D) and two-dimensional swarm structures. Interestingly, the model and stability analysis in [14] are quite similar to the model and proof of stability for the load balancing problem in computer networks [15], [16]. Next, we would note that there have been several investigations into the stability of inter-vehicle distances in "platoons" in intelligent transportation systems (e.g., in [17], or of the "slinky effect" in [18] and [19], and traffic flow in [7]).

This note presents 1-D asynchronous swarm models by putting many single finite-size swarm members together and provide conditions under which swarms will keep their cohesiveness even in the presence of sensing delays and asynchronism. Our desire to consider collision-free cohesion for finite-size vehicles significantly complicates the analysis compared to the case where point-size vehicles are studied and collisions are allowed (e.g., as in [13]). Our study uses a discrete-time discrete event dynamical system [15] approach and unlike the studies of platoon stability in intelligent transportation systems we avoid detailed characteristics of low level "inner-loop control" and vehicle dynamics in favor of focusing on high level mechanisms underlying qualitative swarm behavior when there are imperfect communications. Swarm stability for the $M \geq 2$ dimensional case has been studied in [20] and [21].

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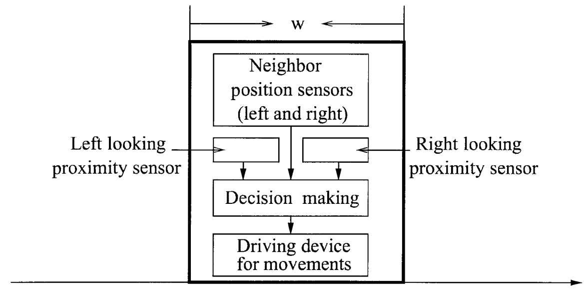

Fig. 1. Single swarm member with a finite size w, 1-D case.

II. Modeling

A 1-D swarm is a set of N swarm members that moves along the real line, where the model of a single swarm member is shown in Fig. 1. Assume it has a physical size (width) $w > 0$ and its position is the center of the square. It has a "proximity sensor" for both sides with a sensing range $\varepsilon > w$, which means that once another swarm member reaches a distance of ε from it, the sensor *instantaneously* indicates the position of the other member. However, if its neighbors are out of its sensing range, the proximity sensor for the left neighbor will return $-\infty$ (or, practically, some large negative number), and the one for the right neighbor will return ∞. The proximity sensor is used to help avoid swarm member collisions and ensures that our framework allows for finite-size vehicles, not just points. It also has a "neighbor position sensor," which can sense positions of neighbors to its left and right if present. Assume that there is no restriction on how close a neighbor must be for the neighbor position sensor to provide a sensed value. The sensed position information may be subjected to random delays. Swarm members like to be close to each other, but not too close. Suppose $d > 0$ is the desired "comfortable distance" between two adjacent neighbors, and $d > \varepsilon$. Each member senses the interneighbor distance via both neighbor position and proximity sensors and makes decisions for movements according to the difference between the sensed distance and the comfortable distance d via its "decision-making" mechanism.

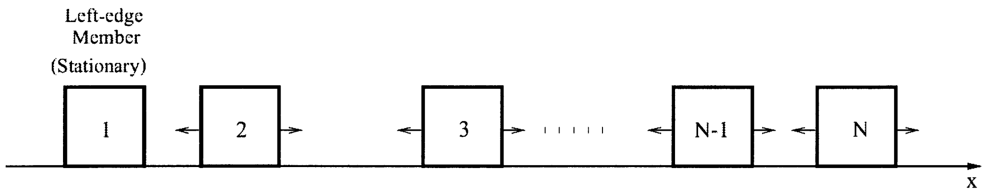

Fig. 2. 1-D asynchronous swarm, all members moving to be adjacent to the stationary edge member.

Then, the decisions are outputted to its "driving device," which provides locomotion for it. Each swarm member tries to move to maintain a comfortable distance to its neighbors. This will tend to make the group move together in a cohesive "swarm."

A 1-D swarm is formed by putting many of the above single swarm members together on the real line as shown in Fig. 2. Let $x^i(t)$ denote the position of swarm member i at time t. We have $x^i(t) \in R$, $i = 1, 2, \ldots, N$, and if $N \geq 2$, we assume that $x^{i+1}(0) - x^i(0) > \varepsilon$, for $i = 1, 2, \ldots, N-1$ initially so that there are no overlapping in finite-size swarm members. Let $x_{pl}^{i-1}(t)(x_{pr}^{i+1}(t))$ denote member i's left-neighbor (right-neighbor) position information sensed by its left-looking (right-looking) proximity sensor at time t. From the previous assumptions, we have

$$x_{pl}^{i-1}(t) = \begin{cases} x^{i-1}(t), & \text{if } x^i(t) - x^{i-1}(t) \leq \varepsilon \\ -\infty, & \text{otherwise} \end{cases} \quad \text{for } i = 2, 3, \ldots, N \tag{1}$$

$$x_{pr}^{i+1}(t) = \begin{cases} x^{i+1}(t), & \text{if } x^{i+1}(t) - x^i(t) \leq \varepsilon \\ \infty, & \text{otherwise} \end{cases} \quad \text{for } i = 1, 2, \ldots, N-1. \tag{2}$$

We assume that every swarm member knows d, and there is a set of times $T = \{0, 1, 2, \ldots\}$ at which one or more swarm members update their positions. Let $T^i \subseteq T$, $i = 1, 2, \ldots, N$, be a set of times at which the ith member's position $x^i(t), t \in T^i$, is updated. Notice that the elements of T^i should be viewed as the indexes of the sequence of physical times at which updates take place, not the real times. These time indexes are nonnegative integers and can be mapped into physical times. The T^i, $i = 1, 2, \ldots, N$, are independent of each other for different i. However, they may have intersections (i.e., it could be that $T^i \cap T^j \neq \emptyset$ for $i \neq j$, so two or more swarm members may move simultaneously). Here, our model assumes that swarm members sense their neighbor positions and update their positions only at time indexes $t \in T^i$ and at all times $t \notin T^i$, $x^i(t)$ is left unchanged. A variable $\tau_{i-1}^i(t) \in T$ (respectively, $\tau_{i+1}^i(t) \in T$) for $i = 2, 3, \ldots, N$ $(i = 1, 2, \ldots, N-1)$ is used to denote the time index of the real time where position information about neighbor $i-1(i+1)$ was obtained by member i at $t \in T^i$ and it satisfies $0 \leq \tau_{i-1}^i(t) \leq t$ $(0 \leq \tau_{i+1}^i(t) \leq t)$ for $t \in T^i$. Of course, while we model the times at which neighbor position information is obtained as being the same times at which one or more swarm members decide where to move and actually move, it could be that the *real time* at which such neighbor position information is obtained is earlier than the real time where swarm members moved. The difference $t - \tau_{i-1}^i(t)(t - \tau_{i+1}^i(t))$ between current time t and the time $\tau_{i-1}^i(t)(\tau_{i+1}^i(t))$ can be viewed as a form of communication delay (of course the actual length of the delay depends on what real times correspond to the indexes t, $\tau_{i-1}^i(t)$, or $\tau_{i+1}^i(t)$). Moreover, it is important to note that we assume that $\tau_{i-1}^i(t) \geq \tau_{i-1}^i(t')$ (respectively, $\tau_{i+1}^i(t) \geq \tau_{i+1}^i(t')$) if $t > t'$ for $t, t' \in T^i$. This ensures that member i will use the most recently obtained neighbor position information. Furthermore, swarm member i will use the real-time neighbor position information $x^{i-1}(t)$ and/or $x^{i+1}(t)$ provided by its proximity sensors instead of information from its neighbor position sensors $x^{i-1}(\tau_{i-1}^i(t))$ and/or $x^{i+1}(\tau_{i+1}^i(t))$ if their neighbors are inside the sensing range of its proximity sensors. This information will be used for position updating until member i gets more recent information, for example, from its neighbor position sensor.

Next, we specify two assumptions to characterize asynchronism for swarms [16].

Assumption 1. (Total Asynchronism): Assume the sets T^i, $i = 1, 2, \ldots, N$, are infinite, and if for each k, $t_k \in T^i$ and $t_k \to \infty$ as $k \to \infty$, then $\lim_{k\to\infty} \tau_j^i(t_k) = \infty$ for $j = i-1, i+1$.

Assumption 2. (Partial Asynchronism): There exists a positive integer B (i.e., $B \in Z^+$, where Z^+ represents the set of positive integers) such that

a) for every i and $t \geq 0$, $t \in T$, at least one of the elements of the set $\{t, t+1, \ldots, t+B-1\}$ belongs to T^i;
b) there holds $t - B < \tau_j^i(t) \leq t$ for all i and $j = i-1, i+1$, and all $t \geq 0$ belonging to T^i.

Notice that in Assumption 1, the delays $t - \tau_j^i(t)$ in obtaining position information of neighbors of member i can become unbounded as t increases. However, Assumption 2 guarantees that the delays $t - \tau_j^i(t)$ in obtaining position information of neighbors of member i is bounded by B and each member moves at least once within B time indexes.

Assume that initially each member i, $i = 1, 2, \ldots, N$, does not have knowledge of its neighbors' positions. Each member remains stationary until it first obtains position information about *both* its neighbors. Then, it will update its position according to this information.

Let $e^i(t) = x^{i+1}(t) - x^i(t)$, $i = 1, 2, \ldots, N-1$ denote the distance between adjacent swarm members. Let the function $g(e^i(t) - d)$ denote the attractive and repelling relationship between two swarm neighbors with respect to the error between $e^i(t)$ and the comfortable distance d. We define two g functions later, $g_a(e^i(t) - d)$ and $g_f(e^i(t) - d)$, to denote two different attractive and repelling relationships that will be used to establish different swarm convergence properties. Assume that for a scalar $\beta > 1$, $g_a(e^i(t) - d)$ is such that

$$\frac{1}{\beta}\left(e^i(t) - d\right) < g_a\left(e^i(t) - d\right) < \left(e^i(t) - d\right), \quad \text{if } \left(e^i(t) - d\right) > 0 \tag{3}$$

$$g_a\left(e^i(t) - d\right) = \left(e^i(t) - d\right) = 0, \quad \text{if } \left(e^i(t) - d\right) = 0 \tag{4}$$

$$\left(e^i(t) - d\right) < g_a\left(e^i(t) - d\right) < \frac{1}{\beta}\left(e^i(t) - d\right), \quad \text{if } \left(e^i(t) - d\right) < 0. \tag{5}$$

Equation (3) indicates an attractive relationship if $(e^i(t) - d) > 0$. In addition, the low bound $1/\beta(e^i(t) - d)$ for $g_a(e^i(t) - d)$ guarantees that swarm member's moving step cannot be infinitely small during its movements to its desired position. The constraint $g_a(e^i(t) - d) < (e^i(t) - d)$ ensures that it will not "over-correct" for the interswarm member distance. Equation (4) indicates a comfortable relationship if $(e^i(t) - d) = 0$. Equation (5) indicates a repelling relationship if $(e^i(t) - d) < 0$.

Assume that for some scalars β and η, such that $\beta > 1$, and $\eta > 0$, $g_f(e^i(t) - d)$ satisfies

$$\frac{1}{\beta}\left(e^i(t) - d\right) < g_f\left(e^i(t) - d\right) < \left(e^i(t) - d\right), \quad \text{if } \left(e^i(t) - d\right) > \eta \tag{6}$$

$$g_f\left(e^i(t) - d\right) = \left(e^i(t) - d\right), \quad \text{if } -\eta \leq \left(e^i(t) - d\right) \leq \eta \tag{7}$$

$$\left(e^i(t) - d\right) < g_f\left(e^i(t) - d\right) < \frac{1}{\beta}\left(e^i(t) - d\right), \quad \text{if } \left(e^i(t) - d\right) < -\eta. \tag{8}$$

These relationships are similar to those for g_a except if $-\eta < (e^i(t) - d) < \eta$, the two swarm members can move to be at the comfortable interswarm member distance within one move.

Assume in Fig. 2 that member 1 remains stationary and all other members calculate their "predicted update step size" by using their left neighbor information and compare this with the right neighbor information obtained from their proximity sensors in order to make movements without collisions at their update time indexes. At $t \in T^i$, the proximity sensors of member i provide its neighbors information $x_{pl}^{i-1}(t)$ and $x_{pr}^{i+1}(t)$ according to (1) and (2) (note that this information may include the real-time position information of its neighbors if its neighbors are inside its ε-range). At the same time, member i also obtains the most recent information of its left neighbor $x^{i-1}(\tau_{i-1}^i(t))$ via its neighbor position sensor. We assume that member i, $i = 2, 3, \ldots, N$, calculates its "predicted update step size" $\phi^i(t)$ according to its left neighbor information at time $t \in T^i$ by (9), as shown at the bottom of page. The update size is computed based on the g function according to the sensed position of member i's left neighbor, where g can be either g_a or g_f, depending on the type of convergence properties we study. The "sgn" function ($\mathrm{sgn}(z) = 1$ if $z \geq 0$, $\mathrm{sgn}(z) = -1$ if $z < 0$) is used to model the moving direction, where "$-$" represents moving to the left and "$+$" represents moving to the right. Member i uses the real-time information of its left neighbor sensed by its proximity sensor in case of $x^i(t) - x_{pl}^{i-1}(t) \leq \varepsilon$, where $x_{pl}^{i-1}(t)$ is from (1). Otherwise, it uses the information $x^{i-1}(\tau_{i-1}^i(t))$ from the neighbor position sensor. Notice that the update step size is forced to be bounded by $(\varepsilon - w)/2$, where $\varepsilon > 0$ is the sensing range of proximity sensors and w is the size of swarm members. With this step size choice, collisions between swarm neighbors with a physical size w can be avoided by proximity sensors even if two swarm members simultaneously move toward each other due to the delayed information. When g_f is used, we assume $\varepsilon - w > 2\eta$, where $\eta > 0$ is a finite positive number so that swarm neighbors can move to be at the comfortable distance d within one move when their distance is already inside η-neighborhood of the comfortable distance.

A mathematical model for the aforementioned swarm, assuming the left-edge member is stationary, is given by

$$\begin{aligned}
x^1(t+1) &= x^1(t) \qquad \forall t \in T^1 \\
x^2(t+1) &= \min\left\{x^2(t) - \phi^2(t), x_{pr}^3(t) - w\right\} \\
&\qquad \forall t \in T^2 \\
\vdots \quad &= \qquad \vdots \\
x^{N-1}(t+1) &= \min\left\{x^{N-1}(t) - \phi^{N-1}(t), x_{pr}^N(t) - w\right\} \\
&\qquad \forall t \in T^{N-1} \\
x^N(t+1) &= x^N(t) - \phi^N(t) \qquad \forall t \in T^N \\
x^i(t+1) &= x^i(t) \qquad \forall t \notin T^i, \qquad i = 1, 2, \ldots, N
\end{aligned} \tag{10}$$

where $\phi^i(t)$ is defined in (9), and $x_{pr}^{i+1}(t)$ is defined in (2). Clearly, in the previous model, swarm member i, $i = 2, 3, \ldots, N-1$, makes decisions for its new position by comparing the right neighbor information obtained from proximity sensors with the predicted position computed from $\phi^i(t)$, where "$\min$" is used to model the avoidance of collisions with a right-neighbor via its right-looking proximity sensor when a swarm member moves to the right (notice that member N moves only according to ϕ^N since it does not have a right neighbor). With this and the choice of initial conditions, we have

$$\left|x^{i+1}(t) - x^i(t)\right| > w, \qquad \text{for } i = 1, 2, \ldots, N-1 \tag{11}$$

since we bound the step size with $(\varepsilon - w)/2$ and the proximity sensor with a sensing range ε is used for collision avoidance. Equation (11) means that swarm members with a finite size w in the aforementioned swarm model have no collisions during movements.

Notice that we assume that member 1 remains stationary, and next we will show how to treat the case where member 1 is mobile. Assume that the left-edge member (member 1) of the swarm in Fig. 2 moves to the left as a edge leader (assume that it does not change directions). All other members will follow it to move to the left. By symmetry, the case where the right-edge member leads the mobile swarm to the right is the same. Assume a swarm member will consider itself to be an edge-member if its neighbor position sensors only indicate the existence of one neighbor, and a middle member if its neighbor position sensors indicate the existence of both left and right neighbors. The edge leader is the left edge member if the swarm moves to the left and is the right edge member if the swarm moves to the right.

For some $\gamma > 0$, we assume $[d - \gamma, d + \gamma]$ is a "comfortable distance neighborhood" relative to $x^i(t)$ and $x^{i+1}(t)$ (i.e., when $x^{i+1}(t) - x^i(t) \in [d - \gamma, d + \gamma]$, we say that they are in the comfortable distance neighborhood), where 2γ is the comfortable distance neighborhood size. Assume that $0 < \varepsilon < d - \gamma$ so that we do not consider swarm member $i+1$ to be at a comfortable distance to member i if it is too close to it, where ε is the sensing range of swarm members' proximity sensors.

Consider the two assumptions of asynchronism. Clearly only Assumption 2 (partial asynchronism) will result in cohesiveness for a

$$\phi^i(t) = \begin{cases} \min\left\{\left|g\left(x^i(t) - x_{pl}^{i-1}(t) - d\right)\right|, \frac{(\varepsilon - w)}{2}\right\} \mathrm{sgn}\left(x^i(t) - x_{pl}^{i-1}(t) - d\right), \\ \quad \text{if } x^i(t) - x_{pl}^{i-1}(t) \leq \varepsilon; \\ \min\left\{\left|g\left(x^i(t) - x^{i-1}\left(\tau_{i-1}^i(t)\right) - d\right)\right|, \frac{(\varepsilon - w)}{2}\right\} \mathrm{sgn}\left(x^i(t) - x^{i-1}\left(\tau_{i-1}^i(t)\right) - d\right), \\ \quad \text{otherwise.} \end{cases} \tag{9}$$

mobile swarm since the delays in Assumption 1 (total asynchronism), which could be unbounded, will make swarm members lose track of their edge leader or their neighbors during movements, i.e., the distance between swarm members could become unbounded just because swarm members use arbitrarily old sensed information. Hence, we will build our model based on Assumption 2 (partial asynchronism) which has a positive integer B. Note that we may view B as an *asynchronism measure*.

For convenience, assume that $x^{i+1}(0) - x^i(0) = d$, for $i = 1, 2, \ldots, N-1$ initially, i.e., at the beginning all the swarm members are at a comfortable distance to their neighbors. Assume the edge leader moves with a step $s(t)$ when $t \geq 0$, where $0 < s(t) \leq r$, i.e., $s(t)$ is bounded by a finite positive number r. So, we have

$$x^1(t+1) = x^1(t) - s(t) \qquad \forall t \in T^1.$$

Member 2 remains stationary until it gets the leader's new position information. Then, this member tries to follow the leader and update its position according to the new position information since it wants to be at the comfortable distance to the edge leader. Similarly, all other swarm members will begin to move and follow their moving neighbors. They all try to be at the comfortable distance to their neighbors. We think of the swarm as maintaining the cohesiveness if all the swarm members are in the comfortable distance neighborhood to their neighbors during the moving process. Notice that the leader's moving step bound r can be used as a measure of how *fast* a cohesive asynchronous swarm moves.

Next, we will show that in the above mobile swarm, when swarm members only update their positions according to the g function, they will never have collisions even without proximity sensors. Accordingly, we define the predicted update step size $\phi^i(t)$ of swarm member i according to its left neighbor information for this case as

$$\phi^i(t) = g\left(x^i(t) - x^{i-1}\left(\tau^i_{i-1}(t)\right) - d\right), \text{ for } i = 2, 3, \ldots, N.$$

Here, "g" can be either g_a or g_f, and it denotes the attractive and repelling relationship between two swarm neighbors. Moreover, we will introduce another such function later and analyze the swarm cohesiveness properties that it achieves.

At the beginning when $t = 0$, $e^i(0) = x^{i+1}(0) - x^i(0) = d$, for $i = 1, 2, \ldots, N-1$. When $t \geq 0$, $t \in T^1$, the edge leader begins to move and we then have $e^1(t) = x^2(t) - x^1(t) > d$. Member 2, the edge leader's right neighbor, remains stationary until it gets the leader's new position information $x^1(\tau^2_1(t))$. From Assumption 2, we know

$$t - B < \tau^2_1(t) \leq t \tag{12}$$

and we have $x^1(\tau^2_1(t)) \geq x^1(t)$. So

$$x^2(t) - x^1\left(\tau^2_1(t)\right) - d \leq x^2(t) - x^1(t) - d. \tag{13}$$

According to the definition of g (i.e., both g_a and g_f) and (13), we have

$$\begin{aligned} g\left(x^2(t) - x^1\left(\tau^2_1(t)\right) - d\right) &\leq x^2(t) - x^1\left(\tau^2_1(t)\right) - d \\ &\leq x^2(t) - x^1(t) - d. \end{aligned} \tag{14}$$

At the beginning, member 2's proximity sensor cannot sense its neighbors since at this time $e^1(t)$ is greater than or equal to d. So, member 2 updates its position only according to $x^1(\tau^2_1(t))$ and the update step is equal to $g(x^2(t) - x^1(\tau^2_1(t)) - d)$. From (14), we know the update step of member 2 is always less than or equal to the error between the real distance from member 2 to the edge leader and d (i.e, $x^2(t) - x^1(t) - d$). Hence, the inter-member distance between members 1 and 2 is always greater than or equal to d. Clearly, a similar result holds for all swarm members, so we have

$$e^i(t) \geq d \qquad \forall t \in T^i, \qquad i = 1, 2, \ldots, N-1. \tag{15}$$

Equation (15) implies that all the mobile swarm members' proximity sensors will never sense their neighbors during movements. This also implies that members will never have collisions in the above swarm, even without proximity sensors. Thus, we can write the following model for the previous mobile swarm:

$$\begin{aligned} x^1(t+1) &= x^1(t) - s(t) \qquad \forall t \in T^1 \\ x^2(t+1) &= x^2(t) - g\left(x^2(t) - x^1\left(\tau^2_1(t)\right) - d\right) \\ &\qquad \forall t \in T^2 \\ \vdots \quad &= \qquad \vdots \\ x^{N-1}(t+1) &= x^{N-1}(t) - g\left(x^{N-1}(t) - x^{N-2}\left(\tau^{N-1}_{N-2}(t)\right) - d\right) \\ &\qquad \forall t \in T^{N-1} \\ x^N(t+1) &= x^N(t) - g\left(x^N(t) - x^{N-1}\left(\tau^N_{N-1}(t)\right) - d\right) \\ &\qquad \forall t \in T^N \\ x^i(t+1) &= x^i(t) \qquad \forall t \notin T^i, \qquad i = 1, 2, \ldots, N. \end{aligned} \tag{16}$$

Remark 1: Notice that for a 1-D N-member collision-free mobile swarm pushed by an edge leader, where member N is the edge leader moving to the left with a step $s(t)$ when $t \geq 0$, $0 < s(t) \leq r$, and $0 < r < \varepsilon - w$, all other members move according to the position of their right neighbor except that they may use position information of their left neighbor for avoiding collisions, Assumption 2 (partial asynchronism) holds, and $x^{i+1}(0) - x^i(0) = d$, for $i = 1, 2, \ldots, N-1$ initially, a mathematical model of the above swarm is given by

$$\begin{aligned} x^1(t+1) &= x^1(t) + \phi^1(t) \qquad \forall t \in T^1 \\ x^2(t+1) &= \max\left\{x^1_{pl}(t) + w, x^2(t) + \phi^2(t)\right\} \\ &\qquad \forall t \in T^2 \\ \vdots \quad &= \qquad \vdots \\ x^{N-1}(t+1) &= \max\left\{x^{N-2}_{pl}(t) + w, x^{N-1}(t) + \phi^{N-1}(t)\right\} \\ &\qquad \forall t \in T^{N-1} \\ x^N(t+1) &= \max\left\{x^{N-1}_{pl}(t) + w, x^N(t) - s(t)\right\} \\ &\qquad \forall t \in T^N \\ x^i(t+1) &= x^i(t) \qquad \forall t \notin T^i, \qquad i = 1, 2, \ldots, N \end{aligned} \tag{17}$$

where the predicted update step size $\phi^i(t)$ of swarm member i according to its right neighbor information is defined as (18) as shown at the bottom of page, and $x^{i-1}_{pl}(t)$ and $x^{i+1}_{pr}(t)$ are defined in (1) and (2). Notice that here we bound the step size by $\varepsilon - w$ instead of $(\varepsilon - w)/2$ since all swarm members only move to the left.

$$\phi^i(t) = \begin{cases} \min\left\{\left|g\left(x^{i+1}_{pr}(t) - x^i(t) - d\right)\right|, \varepsilon - w\right\} \operatorname{sgn}\left(x^{i+1}_{pr}(t) - x^i(t) - d\right), \\ \quad \text{if } x^{i+1}_{pr}(t) - x^i(t) \leq \varepsilon \\ \min\left\{\left|g\left(x^{i+1}\left(\tau^i_{i+1}(t)\right) - x^i(t) - d\right)\right|, \varepsilon - w\right\} \operatorname{sgn}\left(x^{i+1}\left(\tau^i_{i+1}(t)\right) - x^i(t) - d\right), \\ \quad \text{otherwise} \end{cases} \tag{18}$$

III. Convergence Analysis of 1-D Asynchronous Swarms

In this section, we will study stability properties of 1-D asynchronous swarms on the basis of mathematical models we built earlier and provide conditions under which the swarm will obtain and keep the cohesiveness even in the presence of sensing delays and asynchronism. First we will consider stationary edge-member asynchronous N-member swarms, and then we will investigate N-member asynchronous mobile swarms following, or pushed by an edge leader.

A. Convergence Analysis of Stationary Edge-Member Asynchronous Swarms

Here, we will provide conditions under which the swarm in Fig. 2 will converge to be adjacent to a stationary left-edge member. By symmetry, the case for convergence to a stationary right-edge member is the same. We begin with the two-member case, then consider the general N-member case where the proofs will depend on the $N = 2$ case.

Suppose there is a 1-D two-member swarm, which has member i and $i-1$. Assume at the beginning we have $x^i(0) - x^{i-1}(0) > \varepsilon$ and member $i-1$ remains stationary so that proximity sensors of member i will never sense member $i-1$. Therefore, we define the update step size $\phi^i(t)$ of member i in such a two-member swarm as follows:

$$\phi^i(t) = g\left(x^i(t) - x^{i-1}\left(\tau^i_{i-1}(t)\right) - d\right) \tag{19}$$

where the step size only depends on the g function.

Lemma 1: For an $N = 2$ totally asynchronous swarm modeled by

$$\begin{aligned} x^{i-1}(t+1) &= x^{i-1}(t) \qquad \forall t \in T \\ x^i(t+1) &= x^i(t) - \phi^i(t) \qquad \forall t \in T^i \\ x^i(t+1) &= x^i(t) \qquad \forall t \notin T^i \end{aligned} \tag{20}$$

where $x^{i-1}(t) = x_c^{i-1}$, x_c^{i-1} is a constant, $x^i(0) - x^{i-1}(0) > \varepsilon$, and $\phi^i(t)$ is defined in (19) with $g = g_a$, it is the case that for any γ, $0 < \gamma < d - \varepsilon$, there exists a time t' such that $x^i(t') \in [x_c^{i-1} + d - \gamma, x_c^{i-1} + d + \gamma]$ and also $\lim_{t\to\infty} x^i(t) = x_c^{i-1} + d$.

The proof is omitted due to the space limitation. The basic idea is to define a Lyapunov-like function

$$V_i(t) = \left|x^i(t) - \left(x_c^{i-1} + d\right)\right| \qquad \forall t \in T^i$$

and show that for $t \in T^i$, $V_i(t)$ will asymptotically tend to zero via a case analysis using the definition of g_a and Assumption 1.

Corollary 1: For a two-member swarm in Lemma 1, if member $i-1$, instead of remaining stationary at x_c^{i-1}, monotonically approaches the position x_c^{i-1} as $t \to \infty$ (i.e., $x_c^{i-1} \le x^{i-1}(t+1) < x^{i-1}(t)$ or $x^{i-1}(t) < x^{i-1}(t+1) \le x_c^{i-1}, \forall t \in T^{i-1}$), and $|x^{i-1}(0) - x_c^{i-1}| < \gamma$, where $0 < \gamma < d - \varepsilon$, the conclusion of Lemma 1 still holds.

The proof is omitted due to the space limitation. A case study will easily lead to the conclusion.

Lemma 2: For an $N = 2$ partially asynchronous swarm modeled by (20) but with $g = g_f$, where $x^{i-1}(t) = x_c^{i-1}$, x_c^{i-1} is a constant, and $x^i(0) \ge x^{i-1}(0) + \varepsilon$, for any η, $0 < \eta < d - \varepsilon$, swarm member i will converge to the position of $(x_c^{i-1} + d)$ in some finite time, that is bounded by $B[\beta/\eta(|x^i(0) - x_c^{i-1} - d| - \eta) + 2]$.

The proof is omitted due to the space limitation. It is similar to the proof of Lemma 1 and the bound is obtained using the definition of g_f and choosing $\gamma = \eta$.

Theorem 1. (Total Asynchronism, Asymptotic Convergence): For an N-member swarm which is modeled by (10) with $g = g_a$, $N > 1$, Assumption 1 (total asynchronism) holds, and $x^{i+1}(0) - x^i(0) > \varepsilon$, $i = 1, 2, \ldots, N-1$, the swarm members' positions $(x^1, x^2, \ldots, x^N)$ will asymptotically converge to $(x^1(0), x^1(0) + d, x^1(0) + 2d, \ldots, x^1(0) + (N-1)d)$, where $x^1(0)$ is the initial position of the stationary left-edge member.

Proof: We will use a mathematical induction method, where our induction hypothesis will be that k of N swarm members asymptotically converge to $(x^1(0), x^1(0) + d, x^1(0) + 2d, \ldots, x^1(0) + (k-1)d)$ and from this we will show that $k+1$ of N members converge.

First, for $k = 1$, which is the left-edge member, it converges to $x^1(0)$ since it remains stationary. Next we must show that given the induction hypothesis, the first $k+1$ members in the N-member swarm will asymptotically converge to the positions $(x^1(0), x^1(0) + d, x^1(0) + 2d, \ldots, x^1(0) + kd)$.

According to our induction hypothesis we know that there exists a time t^* such that swarm member i, for $i = 2, 3, \ldots, k$, will stay in the range between $x^1(0) + (i-1)d - \gamma$ and $x^1(0) + (i-1)d + \gamma$, where $0 < \gamma < d - \varepsilon$. Assume $\delta_i(t) = x^i(t) - (x^1(0) + (i-1)d)$ for $i = 2, 3, \ldots, k$, which represents member i's distance to its desired position, where $\delta_i(t) \in [-\gamma, \gamma]$ for $t \ge t^*$.

Therefore, after $t \ge t^*$, we have

$$x^i(t+1) = x^1(0) + (i-1)d + \delta_i(t), \qquad \forall t \in T^i, \qquad \text{for } i = 2, 3, \ldots, k$$

for the first k members in the swarm except the stationary member 1 and from member $k+1$ to member N, we have

$$\begin{aligned} x^{k+1}(t+1) &= \min\left\{x^{k+1}(t) - \phi^{k+1}(t), x_{pr}^{k+2}(t) - w\right\} \\ \vdots \quad &= \qquad \vdots \\ x^{N-1}(t+1) &= \min\left\{x^{N-1}(t) - \phi^{N-1}(t), x_{pr}^{N}(t) - w\right\} \\ &\qquad \forall t \in T^{N-1} \\ x^N(t+1) &= x^N(t) - \phi^N(t) \qquad \forall t \in T^N \end{aligned} \tag{21}$$

where g in $\phi^i(t)$ is g_a.

Now, considering member $k+1$'s movements after $t \ge t^*$, we will do a case analysis.

Case 1: If for some $t \in T^{k+1}$, $t \ge t^*$, member $k+1$'s proximity sensor senses member k, i.e.,

$$x_{pl}^k(t) = x^k(t) = x^1(0) + (k-1)d + \delta_k(t)$$

from (9), member $k+1$ will move away from member k according to

$$\begin{aligned} &x^{k+1}(t+1) = \\ &\min\left\{x^{k+1}(t) - \min\left\{\left|g_a\left(x^{k+1}(t) - \left(x^1(0) + (k-1)d + \delta_k(t)\right) - d\right)\right|, \frac{(\varepsilon - w)}{2}\right\}\right. \\ &\quad \cdot \operatorname{sgn}\left(x^{k+1}(t) - \left(x^1(0) + (k-1)d + \delta_k(t)\right) - d\right) \\ &\quad \left. x_{pr}^{k+2}(t) - w\right\} \qquad \forall t \in T^{k+1}. \end{aligned} \tag{22}$$

Case 2: If member $k+1$'s proximity sensor cannot sense member k, it will update its position by

$$\begin{aligned} &x^{k+1}(t+1) = \\ &\min\left\{x^{k+1}(t)\right. \\ &\quad - \min\left\{\left|g_a\left(x^{k+1}(t) - x^k\left(\tau_k^{k+1}(t)\right) - d\right)\right|, \frac{(\varepsilon - w)}{2}\right\} \\ &\quad \left.\cdot \operatorname{sgn}\left(x^{k+1}(t) - x^k\left(\tau_k^{k+1}(t)\right) - d\right), x_{pr}^{k+2}(t) - w\right\} \\ &\forall t \in T^{k+1}. \end{aligned} \tag{23}$$

Case 2a: If member $k+1$ moves toward to member k, due to its outdated information $x^k(\tau_k^{k+1}(t))$ about member k, such that after some time its proximity sensor senses member k, then it will be the same as Case 1.

Case 2b: If member $k+1$ moves according to (23), but its proximity sensor never senses member k, then according to Assumption 1, given time t^*, there exists a time $t^c > t^*$ such that $\tau_k^{k+1}(t) \geq t^*$, $\forall k$ and $t \geq t^c$. So after $t \geq t^c$, member $k+1$ knows member k is at the position $x^1(0)+(k-1)d+\delta_k(t)$. It will move according to (22).

According to our induction hypothesis, we already know

$$\lim_{t\to\infty} \delta_i(t) = 0, \qquad \text{for } i = 2, 3, \ldots k$$

So, if member $k+2$ does not prevent its movements [i.e., $x^{k+1}(t+1)$ is never equal to $x_{pr}^{k+2}(t) - w$ in (22)], as $\delta_k(t)$ goes to zero, member $k+1$ will move to its desired position as $\delta_k(t)$ by

$$\begin{aligned} x^{k+1}(t+1) =& x^{k+1}(t) \\ &- \min\left\{\left|g_a\left(x^{k+1}(t) - \left(x^1(0)+(k-1)d\right.\right.\right.\right. \\ &\left.\left.\left. +\delta_k(t)\right) - d\right)\right|, \frac{(\varepsilon - w)}{2}\right\} \\ &\cdot \operatorname{sgn}\left(x^{k+1}(t) - \left(x^1(0)+(k-1)d+\delta_k(t)\right)\right. \\ &\left. -d\right) \qquad \forall t \in T^{k+1}. \end{aligned} \tag{24}$$

If $|g_a(x^{k+1}(t) - (x^1(0)+(k-1)d+\delta_k(t)) - d)|$ is always less than $(\varepsilon - w)/2$ in (24), member $k+1$ will move by

$$\begin{aligned} x^{k+1}(t+1) =& x^{k+1}(t) \\ &- g_a\left(x^{k+1}(t) - \left(x^1(0)+(k-1)d+\delta_k(t)\right) - d\right) \\ &\forall t \in T^{k+1} \end{aligned}$$

so that it will asymptotically converge to $(x^1(0)+(k-1)d+d) = (x^1(0)+kd)$ by Corollary 1. If $|g_a(x^{k+1}(t) - (x^1(0)+(k-1)d+\delta_k(t)) - d)|$ is larger than $(\varepsilon - w)/2$ member $k+1$ will move to the direction where $|x^{k+1}(t) - (x^1(0)+(k-1)d+\delta_k(t)) - d|$ becomes smaller with a step $(\varepsilon - w)/2$. Therefore, there exists some time t^s that after $t > t^s$, $|g_a(x^{k+1}(t) - (x^1(0)+(k-1)d+\delta_k(t)) - d)|$ is always less than $(\varepsilon - w)/2$ so that it will be the same as before.

If member $k+1$'s proximity sensor finds member $k+2$ nearby at time $t^p \in T^{k+1}$, we have

$$x^{k+1}(t^p+1) = x^{k+2}(t^p) - w.$$

Note that at the same time member $k+2$ also gets member $k+1$'s current position via its proximity sensor since their current adjacent distance is w. There exists a time $t^u \geq t^p+1$, $t^u \in T^{k+2}$ that member $k+2$ will update its position according to

$$\begin{aligned} x^{k+2}(t+1) =& \min\left\{x^{k+2}(t)\right. \\ &- \min\left\{\left|g_a\left(x^{k+2}(t) - \left(x^{k+2}(t^p) - w\right)\right.\right.\right. \\ &\left.\left. -d\right)\right|, \frac{(\varepsilon - w)}{2}\right\} \\ &\cdot \operatorname{sgn}\left(x^{k+2}(t) - \left(x^{k+2}(t^p) - w\right) - d\right) \\ &\left. x_{pr}^{k+3}(t) - w\right\} \qquad \forall t \in T^{k+2}, t \geq t^u. \end{aligned}$$

Member $k+2$'s temporary destination is the position of $(x^{k+2}(t^p) - w + d)$. Similarly, from (21) member $k+3$ could prevent member $k+2$'s further moving, and member $k+4$ could prevent member $k+3$, etc. In the end, member N could prevent member $N-1$. However, member N is free to move to the right.

Since we assume the sets T^i are infinite, and each swarm member moves infinitely often, we can easily use another induction method to show that in the previous cases after some time, member N could move away from member $N-1$, and member $N-1$ could move away from member $N-2$, etc. So, after member $k+2$ moves away from member $k+1$, member $k+1$ will continue moving to the position $(x^1(0)+kd)$. However, member $k+2$ may prevent member $k+1$ again after $t > t^u$ due to the total asynchronism. It is similar for member $k+3$, $k+4$, and so on. In that case, it will repeat the aforementioned process. As we know member $k+2$'s distance to the position $(x^1(0)+kd)$ is finite, and there are only a finite number of members in the swarm, and swarm member's movements cannot be infinitely small if the distance to its desired position is not infinitely small via the definition of g_a. So there exists a time $t^{np} > t^p$ such that member $k+2$ will have moved beyond the position $(x^1(0)+kd)$. After $t > t^{np}$, member $k+2$ will never prevent member $k+1$ again. From Corollary 1 and our induction hypothesis, member $k+1$ will asymptotically converge to the position $(x^1(0)+kd)$. This ends the induction step. **Q.E.D.**

Theorem 2. (Partial Asynchronism, Finite Time Convergence): For an N-member swarm which is modeled by (10) with $g = g_f$, $N > 1$, Assumption 2 (partial asynchronism) holds, and $x^{i+1}(0) - x^i(0) > \varepsilon$, $i = 1, 2, \ldots, N-1$, for any η, $0 < \eta < (\varepsilon - w)/2$, the swarm members' positions $(x^1, x^2, \ldots, x^N)$ will converge to $(x^1(0), x^1(0)+d, x^1(0)+2d, \ldots, x^1(0)+(N-1)d)$ in some finite time, that is bounded by

$$B2^{N-2}\left[\frac{\beta}{\eta}\left(\max_i\left(\left|x^i(0) - x^1(0) - (i-1)d\right|\right) - \eta\right) + 2\right] \tag{25}$$

for $i = 2, 3, \ldots, N$, where $x^1(0)$ is the initial position of the stationary left-edge member.

Proof: The proof is similar to Theorem 1 except that the swarm is partially asynchronous. So, we will use Assumption 2 and Lemma 2 to deduce member $k+1$ will arrive at the position $(x^1(0)+kd)$ in some finite time.

Next, we will try to bound the amount of converging time for a N-member swarm. From Lemma 2, for a two-member swarm the time needed to achieve convergence is bounded by $B[\beta/\eta(|x^i(0) - x_c^{i-1} - d| - \eta) + 2]$. Similarly for a N-member swarm, if blockades never occur among swarm members, the total converging time is bounded by $BN[\beta/\eta(\max_i(|x^i(0) - x^1(0) - (i-1)d|) - \eta) + 2]$, for $i = 2, 3, \ldots, N$. On the other hand, if all middle swarm members are blocked by their neighbors in every step and cannot move until the blockades are released, there are $N-2$ members preventing member 2, $N-3$ members preventing member 3, and so on, until only one member prevent member $N-1$. As we know $(N-2)+(N-3)+(N-4)+\cdots+1 < 2^{N-2}$. Moreover, we know each middle swarm member's distance to its desired position $|x^i(0) - x^1(0) - (i-1)d|$ is finite and each of their moving steps is at least larger than η/β. Therefore, the maximum number of total update time steps is $2^{N-2}[\beta/\eta(\max_i(|x^i(0) - x^1(0) - (i-1)d|) - \eta) + 2]$. Then we can bound the total converging time by (25) in this worst case because the maximum update time interval is B. **Q.E.D.**

Note that some proof details of Theorems 1 and 2 are omitted due to the space limitation. Interested readers can refer to [22].

B. Convergence Analysis of Asynchronous Mobile Swarms

Next, we will study cohesiveness of a mobile swarm following an edge leader. First, we will study the case of using the g_f function (assuming $\gamma = \eta$), and then show that all members in an N-member mobile swarm will be in a comfortable distance neighborhood from their neighbors during movements if there are constraints on the leader's moving step bound, the partial asynchronism measure, and the comfortable distance neighborhood size. While we only consider a left-edge member leading a swarm to the left, by symmetry, the case where the right-edge member leads the mobile swarm to the right is the same (only the model is different).

Theorem 3: For an N-member asynchronous mobile swarm modeled by (16), where g is g_f, $N > 1$, Assumption 2 (partial asynchronism) holds, and $x^{i+1}(0) - x^i(0) = d$, $i = 1, 2, \ldots, N-1$, if

$$0 < r \leq \frac{\gamma}{NB - 1} \tag{26}$$

for a given γ, all the swarm members will be in the comfortable distance neighborhood $[d, d+\gamma]$ of their neighbors during the moving process, where r is the upper bound of the edge leader's moving step $s(t)$, $B \in Z^+$ is the partial asynchronism measure and γ (choose $\gamma = \eta$) is the comfortable distance neighborhood size.

The proof is omitted due to the space limitation. Interested readers are welcome to contact the authors to get it or refer to [23].

Remark 2: Note the following about (26).

- For a given B and γ, it provides bound on how fast a N-member swarm can move and still maintain the type of cohesiveness characterized by γ. For example, increases in swarm size, communication delays, or swarm cohesiveness (smaller γ) require decreases in the rate of movement of the leader.
- For a given r and B, it provides the size of the neighborhood that will be maintained and hence specifies a degree of cohesiveness of a N-member swarm.
- For a given r and γ, it provides constraints on how to design a communication system (i.e., what is needed for B) for a N-member swarm between swarm members and indicates how often they must update their positions.

Remark 3: From Theorems 1 and 2, we can see that if the edge leader stops moving (i.e., $s(t) = 0, t \geq t^1$ for some $t^1 \in T^1$), all other $N-1$ members will converge to be adjacent to the edge leader with a comfortable distance d.

Remark 4: If a different g_b function is defined and used in Theorem 3, which does not require a swarm member to move to be adjacent to its neighbor in one step if it gets very close to it but guarantee that the following members are in the b-neighborhood of desired comfortable distance of their leading neighbors after each update step (the definition of g_b function is omitted due to the space limitation), with all other same conditions, and if

$$0 < r \leq \frac{\gamma - b}{NB - 1} \tag{27}$$

we can get the same conclusion as in Theorem 3.

Next, we provide different conditions under which an N-member asynchronous mobile swarm pushed by an edge leader can maintain cohesiveness during movements using the model in (17). While we only consider a right-edge member pushing a swarm to the left, by symmetry, the case where the left-edge member pushes the mobile swarm to the right is the same.

Theorem 4: For an N-member asynchronous mobile swarm modeled by (17), where g is g_f, $N > 1$, Assumption 2 (partial asynchronism) holds, $0 < \gamma < d - \varepsilon$, and $x^{i+1}(0) - x^i(0) = d$, $i = 1, 2, \ldots, N-1$, if

$$0 < r \leq \frac{\gamma}{NB - 1} \tag{28}$$

for a given γ, all the swarm members will be in the comfortable distance neighborhood $[d-\gamma, d]$ of their neighbors during the moving process, where r is the upper bound of the edge leader's moving step $s(t)$, $B \in Z^+$ is the partial asynchronism measure, γ (choose $\gamma = \eta$, and $0 < \eta < \varepsilon - w$) is the comfortable distance neighborhood size, and ε is the sensing range of proximity sensors.

The proof is omitted due to the space limitation. It is similar to the proof of Theorem 3.

Remark 5: If g_b function described in Remark 5 is used in Theorem 4 with all other same conditions, and if

$$0 < r \leq \frac{\gamma - b}{NB - 1} \tag{29}$$

we can get the same conclusion as in Theorem 4.

IV. Conclusion

We have provided results for a 1-D swarm showing how to achieve cohesion and collision avoidance even with delays, asynchronism, and finite body sizes. In the future we are going to consider sensor dynamics and noise, vehicle dynamics, network topology impacts, and the higher dimensional case. Some progress on the last two topics is given in [20], [21], and [24].

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