Stable reactive power balancing strategies of grid-connected photovoltaic inverter network

Zhongkui Wang[*,†] and Kevin M. Passino

Department of Electrical and Computer Engineering, The Ohio State University, Columbus, OH, USA

SUMMARY

In this paper, a distributed reactive power control based on balancing strategies is proposed for a grid-connected photovoltaic (PV) inverter network. Grid-connected PV inverters can transfer active power at the maximum power point and generate a certain amount reactive power as well. Because of the limited apparent power transfer capability of a single PV inverter, multiple PV inverters usually work together. The communication modules of PV inverters formulate a PV inverter network that allows reactive power to be cooperatively supplied by all the PV inverters. Hence, reactive power distributions emerge in the grid-connected PV inverter network. Uniform reactive power distributions and optimal reactive power distributions are considered here. Reactive power balancing strategies are presented for both desired distributions. Invariant sets are defined to denote the desired reactive power distributions. Then, stability analysis is conducted for the invariant sets by using Lyapunov stability theory. In order to validate the proposed reactive power balancing strategies, a case study is performed on a large-scale grid-connected PV system considering different conditions. Copyright © 2015 John Wiley & Sons, Ltd.

1. INTRODUCTION

In the alternating-current (AC) power grid, the phase difference between voltage and current leads to the occurrence of reactive power. Reactive power serves the important role of maintaining voltage levels and accomplishing the transmission of active power in the power grid [1, 2]. Control and optimization techniques for reactive power generation, absorption, and flow in existing power grids have been given significant attention [3]. The current power grid is developing into a smart grid with fault-tolerant, self-monitoring, and self-healing capabilities to intelligently deal with generation diversification, optimal deployment of expensive assets, demand response, energy conservation, and so on [4]. The application of distributed generation (DG), such as grid-connected photovoltaic (PV) systems, will also be increasingly used in the smart grid. At the end of 2013, the worldwide total capacity of installed solar PV systems reached 139 GW [5] of which a large portion is grid-connected PV systems [6]. In grid-connected PV systems, DC–AC inverters are used to transfer active power generated by PV panels to the grid. The power rating of a PV inverter is usually from 10 to 500 kW. In large-scale grid-connected PV systems, for instance, solar farms with MW-scale ratings, multiple PV inverters are connected in parallel to satisfy the requirement of transferring a large amount of power [7–9].

Although certain standards [10] do not permit inverter-based DGs to regulate local voltage currently, in the future smart grid, reactive power will also be provided by DGs with smart inverters. A variety of literature such as [11, 12] addresses the control and optimization problems of reactive

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*Correspondence to: Zhongkui Wang, Department of Electrical and Computer Engineering, The Ohio State University, 2015 Neil Ave, Columbus, OH 43210, USA.

[†]E-mail: wang.1231@osu.edu

power for grid-connected PV systems with a single DC–AC inverter. In [11], several reactive power control methods and different PV inverters with modes to support reactive power have been compared. In [12], an online optimal control strategy to minimize the energy losses of grid-connected PV inverters is proposed. Research efforts have been provided for multiple PV inverters as well, such as the reactive power optimization of multiple PV generators in a distribution network [13–16]. These optimization problems deal with the minimization of either voltage deviation of, or the power loss between, distribution feeders. An adaptive control scheme is developed to solve the problem in [13], and numerical methods are employed in [14–16]. For multiple inverters collocated in parallel, the "droop control" method is widely used for load sharing [17, 18]. Droop control basically determines the load of each inverter based on the power rating and the slope of droop characteristics. However, droop control does not have much flexibility to deal with different control and optimization purposes. In large-scale grid-connected PV systems, nonuniform solar irradiation across the whole system and tight apparent power limits of each inverter call for more sophisticated control. As indicated in [19], the components of the future smart grid will have independent processors and have the capability to cooperate and compete with others. Some smart PV inverters have communication modules installed, and a PV inverter network can be established to allow the application of advanced control and optimization techniques.

Our work in this paper focuses on a distributed reactive power control strategy for a PV inverter network. It proposes an approach that involves reactive power allocation across the PV inverters for a variety of control purposes. Typically, each inverter has a classical pulse-width-modulation controller with an inner current loop and an outer voltage loop, both using proportional-integral controllers. In this control structure, the oscillating current and voltage in the *abc* frame are transformed into a *direct-quadrature* reference frame. Then, the setpont of active and reactive power can be controlled by using the quantities in *d-q* frame. However, detailed control schemes for individual inverters are beyond the scope of this work, and we assume that each inverter is capable of controlling itself properly for given active and reactive power set points.

The distributed reactive power control in this paper is based on the balancing strategies that are similar to [20–24]. In [20, 21], load balancing strategies are adopted for a computer processor network to balance the tasks being processed. Similarly, task load balancing strategies are used by networked autonomous air vehicles in [22]. In [23, 24], balancing strategies are designed to achieve certain desired distributions of multiple agents among "habitats." Because of the balancing strategies-based distributed control, all the individuals in the system are networked and can cooperatively work together without a higher-level controller. This technique is also applicable for the reactive power control of the grid-connected PV inverter network. Inverters in the network can communicate with, and "pass reactive power to," each other to either alleviate the stress of certain inverters or achieve any desired reactive power distribution. In Section 2, the system model, including the grid-connected PV system model and the communication network model, is presented. Section 3.1 provides the reactive power "passing strategies" for different desired reactive power distributions in the PV inverter network. These desired reactive power distributions are represented by certain invariant sets, and these invariant sets are proven to be stable by using the Lyapunov stability theory. Then, the balancing strategy-based reactive power control for the grid-connected PV inverter network is tested against a sample PV inverter network in Section 4. Simulation results are shown for different initial conditions and desired reactive power distributions. The impact of topology differences for the PV inverter network is evaluated in simulations as well. Finally, conclusions are provided in Section 5.

2. THE SYSTEM MODEL

We first specify a system model for the PV inverter network in large-scale grid-connected PV systems. The system model is decentralized as the DC–AC inverters are separate entities that have certain autonomy to regulate local active and reactive power generation, and communications among the PV inverters allow them to formulate an inverter network. The entire model is in a discrete time framework, and we assume all PV inverters use the same global time reference. We also assume that

the dynamics and local control of individual PV inverters are much faster than the control for the entire system. By such an assumption, we consider the inverters as multiple nodes in the network, and we focus on the balancing strategies for the reactive power control of the overall system.

2.1. The grid-connected photovoltaic systems

Consider a grid-connected PV system with $N \in \mathbb{N}_+$ PV inverters that form an inverter network. There are a considerable number of PV panels that are attached to each inverter. These PV panels are usually connected together into a PV string then to the PV inverter. The system diagram is shown in Figure 1. Let the continuous variable $x_i \in \mathbb{R}$, $i \in \{1, \ldots, N\}$ be the amount of reactive power of the ith PV inverter and $P_i \in \mathbb{R}_+$ be the amount of active power transferred by the same PV inverter. Suppose that $\sum_{i=1}^{N} x_i = Q_D$, where Q_D is the reactive power demand from the utility grid, which is known, but it could be time-variant (i.e., the reactive power demand of the grid varies for different times of the day). Here, we denote that positive Q_D is the reactive power that inverters supply to the grid, and negative Q_D is the reactive power that inverters absorb from the grid. Also, we assume the same sign convention for the reactive power of individual inverter x_i. The ith PV inverter has limited capability to transfer active power and generate reactive power. We still use a constant $C_i > 0$ to represent the current limit of the ith inverter. The value of $C_i > 0$ is optimally designed based on the rating of the active power of the ith inverter. This implies a trade-off between the inverter cost and the power transfer capability of the ith inverter. As the current of the ith PV inverter is not allowed to exceed C_i, we have

$$C_i - \frac{\sqrt{P_i^2 + x_i^2}}{3|V|} \geqslant 0 \implies$$
$$-\sqrt{9|V|^2 C_i^2 - P_i^2} \leqslant x_i \leqslant \sqrt{9|V|^2 C_i^2 - P_i^2}, \ i = 1, \ldots, N \tag{1}$$

where $|V|$ represents the magnitude of the grid line-to-neutral voltage and is known. Then $q_i^{\max} = \sqrt{9|V|^2 C_i^2 - P_i^2}$ and $q_i^{\min} = -\sqrt{9|V|^2 C_i^2 - P_i^2}$ are the upper and lower bounds of x_i. As we will present reactive power balancing strategies, we use a discrete time formulation. Hence, we use $x_i(k)$ to denote the reactive power x_i at time k.

2.2. Communication network

We adopt a communication network for the DC–AC inverters of the grid-connected PV systems that is similar to the ones for the systems given by [21] and [23]. There are different candidate
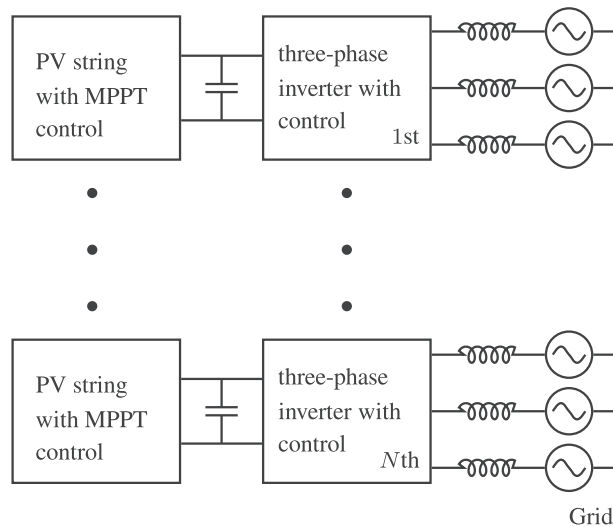

Figure 1. System diagram of the photovoltaic (PV) inverter network in grid-connected PV systems.

topologies for the communication system of the DC–AC inverters (i.e., line, ring, and network). We assume that the communication links and the topology are fixed. Also, we assume that the communication links have sufficient capacity to transmit the required information, and the only deficiency is a possible delay that occurs during the sensing process and information transmission. We assume that the local control of the PV inverters' dynamics is fast enough so that the possible delays due to the local control operation are negligible. The communication network of these PV inverters $\mathcal{I} = \{1, 2, \ldots, N\}$ is described by a directed graph $\mathcal{G} = (\mathcal{I}, \mathcal{A})$, where $\mathcal{I}$ represents the DC–AC inverters in the network that we assume to be nodes, and $\mathcal{A} = \{(i, j) : i, j \in \mathcal{I}\}$ is a set of directed arcs that represents the communication links and $\mathcal{A} \subset \mathcal{I} \times \mathcal{I}$. For each $i \in \mathcal{A}$, there must exist $(i, j) \in \mathcal{A}$ such that each DC–AC inverter is guaranteed to be connected to the network, and if $(i, j) \in \mathcal{A}$, then $(j, i) \in \mathcal{A}$. We assume that $(i, i) \notin \mathcal{A}$, as each inverter does not need to communicate with itself and does not balance reactive power with itself.

3. STABLE DISTRIBUTED REACTIVE POWER CONTROL BASED ON BALANCING STRATEGIES

According to the operation mode, the balancing strategy varies. Here, we propose different balancing strategies via different operation modes and objectives. We will prove the balancing strategies are stable with respect to an invariant set that represents the desired reactive power distribution. First, we consider the case where the reactive power is uniformly balanced among the PV inverters. Such a balancing strategy is able to alleviate the stress of each PV inverter and can be applied in the night operation mode. Then, the balancing strategy for optimal reactive power distribution is derived by modifying the balancing strategy for uniform reactive power distribution.

3.1. Uniformly distributed reactive power

Because of the limited capability of a single PV inverter, the amount of reactive power that one inverter can generate under certain active power transferred, and certain power factor is bounded below (capacitive reactive power) and above (inductive reactive power). Without loss of generality, we define $q_i^{\min}$ and $q_i^{\max}$ to be the minimum and maximum reactive power that inverter i can generate, respectively, and assume that $q^{\min} < 0$ and $q_i^{\max} > 0$. Notice that $q_i^{\max}$ and $q_i^{\min}$ are not necessarily time-invariant, that is, their values can change when the environmental conditions of inverter i change. Hence, we use $q_i^{\max}(k)$ and $q_i^{\min}(k)$ to denote the upper and lower bounds of the reactive power of inverter i at time k. We focus on a simplex $\Delta = \{x \in \mathbb{R}^N : \sum_{i=1}^N x_i = Q_D\}$ in which the x_i dynamics evolve, where we assume that Q_D is constant and known. Time-varying Q_D only changes the initial conditions of the reactive power distribution, and the reactive power passing strategies for this case (which are similar to the ones we will be proposing) are not considered. Here, we let $\mathcal{U}(k) = \{i \in \mathcal{I} : q_i^{\min}(k) < x_i < q_i^{\max}(k)\}$ represent the set of inverters with unsaturated reactive power, that is, in which the reactive power does not reach the bounds at time k, and let $\mathcal{S}(k) = \mathcal{I} - \mathcal{U}(k)$ represent the set of inverters with saturated reactive power, that is, in which the reactive power reaches the bounds. We present a class of reactive power passing strategies considering the reactive power bounds of inverters. Then, a distribution of reactive power is presented by an invariant set and proved to be stable in the sense of Lyapunov under certain conditions.

3.1.1. Reactive power passing strategies.
Let $x_i(k)$ be the reactive power of inverter i at time k. For any $(i, j) \in \mathcal{A}$, let $x_j^i(k)$ be the amount of reactive power of inverter j that inverter i perceives at time k. It is the reactive power information sent to inverter j from i. Define $\alpha_i^{i \to j}$ to be the amount of reactive power that inverter i passes to inverter j. By saying *reactive power passing*, we mean that $\alpha_i^{i \to j}$ is the amount of reactive power removed from inverter i when i passes to inverter j, and it is also the amount of reactive power that adds to inverter j. Define $\alpha_j^{i \to j}$ as the amount of reactive power received by inverter j due to inverter i sending reactive power to j at time k. As the total desired reactive power Q_D can be both inductive (positive) and capacitive (negative), we assume that when the inverters are balancing a total amount of inductive reactive power, the reactive power

being passed between inverters is capacitive , that is, when $Q_D > 0$, we assume that $\alpha_i^{i \to j} < 0$. For the case where the inverters are balancing a capacitive reactive power, we assume that the reactive power being passed between inverters is inductive (positive), that is, when $Q_D < 0$, we assume that $\alpha_i^{i \to j} > 0$. Let $\mathcal{N}_i = \{j : (i,j) \in \mathcal{A}\}$ be the subset of the neighboring nodes of inverter i. Then, the following conditions define a class of reactive power passing strategies for inverter i at time k with the considerations of inverter bounds. When $Q_D > 0$, we assume that inverters can only pass capacitive reactive power between each other.

(a1) $\alpha_i^{i \to j} = 0$, if $x_i(k) - x_j^i(k) \geqslant 0$ or if $x_i(k) = q_i^{\max}(k)$;

(a2) $x_i(k) - \sum\limits_{\{j : j \in \mathcal{N}_i\}} \alpha_i^{i \to j} \leqslant \min\{x_j^i(k) + \alpha_i^{i \to j}, q_i^{\max}(k) + \alpha_i^{i \to j}\}$, $\forall\, j \in \mathcal{N}_i$ such that $x_i(k) - x_j^i(k) < 0$;

(a3) If $\alpha_i^{i \to j} < 0$ for some j, then $\alpha_i^{i \to j*} \leqslant \gamma_{ij*} \max\left\{\left[x_i(k) - x_{j*}^i(k)\right], \left[x_i(k) - q_i^{\max}\right]\right\}$ for some $j^* = \arg\max\limits_{j'} \left\{x_{j'}^i(k) : j' \in \mathcal{N}_i\right\}$;

where $\gamma_{ij} \in (0, 1)$ for $j \in \mathcal{N}_i$ is a constant that represents the proportion of reactive power difference that inverters try to reduce by passing from inverter i to j. The conditions of $Q_D < 0$ are symmetric to the conditions of $Q_D > 0$ that are not presented here.

Condition (a1) indicates that inverter i will not pass any capacitive reactive power to its neighboring inverter j if its reactive power perception about inverter j is greater than its own reactive power, that is, if the reactive power i is greater than the reactive power perception of inverter j, inverter i will not increase the reactive power level of itself to decrease the reactive power level of inverter j. Also, inverter i will not pass any capacitive reactive power to its neighboring inverters if the reactive power of inverter i reaches the upper bound, that is, inverter i cannot take more inductive reactive power for this case. Condition (a2) limits the amount of capacitive reactive power that inverter i can pass to its neighbor nodes then limits the increase of the reactive power level of inverter i. It indicates that after the reactive power transfer, the reactive power of inverter i must be not higher than the reactive power perception of any of its neighbor inverters or its upper bound. This condition excludes the oscillation of reactive power between inverters. Condition (a3) implies that if inverter i passes some capacitive reactive power to its neighboring nodes, then it must pass some non-negligible amount of capacitive reactive power to the neighboring inverter with maximum reactive power level. Meanwhile, the reactive power of inverter i is guaranteed not to exceed the upper bound.

3.1.2. Distribution of reactive power. The state equation of x_i with the reactive power passing strategies presented previously is

$$x_i(k+1) = x_i(k) - \sum_{\{j : (i,j) \in \mathcal{A}\}} \alpha_i^{i \to j} + \sum_{\{j : (i,j) \in \mathcal{A}\}} \alpha_i^{j \to i}, \ \forall i \in \mathcal{I} \tag{2}$$

Let $\mathcal{X} = \Delta$ be the set of states and $x(k) = [x_1(k), \ldots, x_N(k)]^\top \in \mathcal{X}$ be the state vector, with $x_i(k)$ the reactive power of inverter i at time $k \geqslant 0$. Then, the set

$$\begin{aligned}
\mathcal{X}_c = \Big\{x \in \mathcal{X} : &\text{ for all } i \in \mathcal{I}, \text{ either } x_i(k) = x_j(k) \text{ for all } (i,j) \in \mathcal{A} \text{ such that}\\
&q_i^{\min}(k) < x_j(k) < q_i^{\max}(k), \ x_i(k) > x_j(k) \text{ for all } (i,j) \in \mathcal{A} \text{ such that}\\
&x_j(k) = q_j^{\max}(k), \text{ and } x_i(k) < x_j(k) \text{ for all } (i,j) \in \mathcal{A} \text{ such that}\\
&x_j(k) = q_j^{\min}(k); \text{ or } x_i(k) = q_i^{\max}(k) \text{ or } q_i^{\min}(k)\Big\}
\end{aligned} \tag{3}$$

represents a distribution of the reactive power on the inverter network. Any distribution $x \in \mathcal{X}_c$ is such that for any $i \in \mathcal{I}$ either $x_i = q^{\max}(k)$ when $Q_D > 0$, $x_i = q^{\min}(k)$ when $Q_D < 0$; or if $q^{\min}(k) < x_i < q^{\max}(k)$, it must be the case that all neighboring inverters $j \in \mathcal{N}_i$ such that $q_j^{\min}(k) < x_j < q_j^{\max}(k)$ have the same reactive power levels as inverter i. In $\mathcal{X}_c$, if $x_j = q_j^{\max}(k)$ when $Q_D > 0$ for $j \in \mathcal{N}_i$, then $x_j \leqslant x_i$; if $x_j = q_j^{\min}(k)$ when $Q_D < 0$ for $j \in \mathcal{N}_i$, then

$x_j \geqslant x_i$. Notice that when $x(k) \in \mathcal{X}_c$, there is only one reactive power passing strategy that satisfies conditions (a1)–(a3), that is, $\alpha_j^{i \to j} = 0$ for all $i \in \mathcal{I}$. Recall that for all $x \in \mathcal{X}$, there exists a subset of inverters with unsaturated reactive power, denoted by $\mathcal{U}(k)$. For any $x \in \mathcal{X}_c$, the subset $\mathcal{U}(k)$ is not unique, and the specific equalized reactive power levels of inverters in this subset are not always known as *a priori* or at any point before the set $\mathcal{X}_c$ is achieved. The set $\mathcal{U}(k)$ and the equalized reactive power levels emerge while the reactive power is distributed over the inverters.

3.1.3. Emergence of inverter islands. According to the definition of $\mathcal{X}_c$, it is possible that inverters in the subset $\mathcal{U}(k)$ are isolated (by the inverters with saturated reactive power) and have different reactive power levels. This could occur, for instance, if two inverters with high reactive power levels are separated by an inverter with saturated reactive power, that is, $x_{i-1}(k) \neq x_{i+1}$, $x_i(k) = q_i^{\max}(k)$, and $x_i(k) \leqslant \min\{x_{i-1}(k), x_{i+1}(k)\}$. Hence, depending on the graph's topology, there could be isolated "islands" of inverters of which the reactive power does not reach the bounds, where only inverters belong to the same island have the same reactive power level. Moreover, notice that the formation of inverter islands depends on the total reactive power, their initial distribution $x(0)$, and the changes of environmental conditions, that is, $q_i^{\max}(k)$ and $q_i^{\min}(k)$.

3.1.4. Stability analysis. Let us consider the reactive power distribution defined by Equation (3). As discussed previously, the invariant set consists of many elements that represent different reactive power distributions, and some distributions can lead to saturated reactive power on certain inverters (i.e., reactive power of that inverter hits the bounds). The next theorem shows that under certain situations, there is no inverter with saturated reactive power, and the distribution represented by the invariant set is unique.

Lemma 1 (Uniform distribution, unsaturated reactive power, uniqueness of invariant set)
If $q_i^{\max}$ and $q_i^{\min}$ are consistent with time k and the total amount reactive power satisfies $N \max_i\{q_i^{\min}\} < Q_D < N \min_i\{q_i^{\max}\}$, then the invariant set $\mathcal{X}_c$ satisfies $|\mathcal{X}_c| = 1$, and the invariant set $\mathcal{X}_c$ is simplified to $\mathcal{X}_c = \{x \in \mathcal{X} : \text{for all } i \in \mathcal{I}, \ x_i(k) = x_j(k) \text{ for all } (i, j) \in \mathcal{A}\}$.

Proof
See Appendix A. □

Lemma 1 implies the conditions under which there is no inverter with saturated reactive power (i.e., no inverter's reactive power hits the bounds) and the uniqueness of the invariant set. All inverters will eventually have the same reactive power level, and the reactive power level only depends on the number of inverters N and the total desired reactive power Q_D. Then, the following analysis considering inverter reactive power bounds is restricted to the following scenario:

Assumption 1 (Complete graph, consistent inverter constraints, saturated reactive power)
   (i) The graph $\mathcal{G} = (\mathcal{I}, \mathcal{A})$ is fully connected.
  (ii) The environmental conditions of inverters are consistent, that is, $q_i^{\max}$ and $q_i^{\min}$ are time-invariant for all $i \in \mathcal{I}$.
 (iii) The total amount reactive power Q_D satisfies either $N \min_i\{q_i^{\max}\} < Q_D < \sum_{i=1}^{N} q_i^{\max}$ for $Q_D > 0$ or $\sum_{i=1}^{N} q_i^{\min} < Q_D < N \max_i\{q_j^{\min}\}$ for $Q_D < 0$.

Assumption 1 (i) indicates a complete graph in which every node is connected to other nodes; (ii) guarantees the bounds of reactive power for each inverter are fixed and known; (iii) is the condition such that the reactive power of certain inverters in the network will eventually reach either the lower bound or upper bound, but Q_D is less than the greatest total reactive power capability of the entire system.

Lemma 2 (Complete graph, uniqueness of invariant set)
With conditions (i) and (ii) of Assumption 1 and any total amount of reactive power such that $\sum_{i=1}^{N} q_i^{\min} < Q_D < \sum_{i=1}^{N} q_i^{\max}$, the invariant set $\mathcal{X}_c$ satisfies $|\mathcal{X}_c| = 1$.

Proof
See Appendix A. □

Lemma 2 implies that for a fully connected graph topology, there are no isolated inverters with different reactive power levels. The full connectivity of the inverters leads to reactive power equalization across all inverters with unsaturated reactive power and the emergence in some cases (i.e., the cases given by condition (iii) of Assumption 1) of a set of inverters with saturated reactive power.

Lemmas 1 and 2 studied the characteristics of the invariant set $\mathcal{X}_c$ that represents the reactive power distribution for different total reactive power amount and connectivity topologies of the network. We now focus on the analysis of inverters approaching this set especially for the case that some inverters have saturated reactive power.

Theorem 1 (Complete graph, emergence of saturated inverters, asymptotic stability in large)
With Assumption 1 and the reactive power passing strategies (a1)–(a3), the invariant set $\mathcal{X}_c$ is asymptotically stable in large.

Proof
See Appendix A. □

Theorem 1 considers the inverters with saturated reactive power and studies the stability properties of the invariant set. With the reactive power passing conditions (a1)–(a3), Theorem 1 indicates on a complete graph for the total reactive power Q_D that satisfies Assumption 1 the reactive power distribution will eventually end in the invariant set $\mathcal{X}_c$, that is, the reactive power of some inverters is saturated at the bounds and other inverters equalize the reactive power level. We now assume more restrictive reactive power passing conditions in order to study the rate of convergence to the desired distribution. In particular, we assume

Assumption 2 ((Rate of occurrence))
Every B time steps, there is the occurrence of the reactive power passing behaviors that are defined by conditions (a1)–(a3) for every inverter.

Then, the following theorem is derived.

Theorem 2 (Complete graph, emergence of saturated inverters, exponential stability)
With Assumptions 1 and 2, and the reactive power passing strategies defined by conditions (a1)–(a3), the invariant set $\mathcal{X}_c$ is exponentially stable in large.

Proof
See Appendix A. □

3.2. Optimally distributed reactive power

Multiple capability-limited inverters in the network can cooperatively generate a large amount of desired reactive power for the grid with the uniformly distributed reactive power balancing conditions. Such conditions aim at achieving an equalized reactive power level on all inverters in the system. Under some circumstances, the inverter constraints confine the reactive power of certain inverters below the the equalized reactive power level of others when $Q_D > 0$ (above the equalized reactive power level of others when $Q_D < 0$). We now modify the reactive power balancing conditions to consider the optimality of the allocation of reactive power on inverters. We focus on an optimally allocated reactive power profile that can achieve a maximum total "safety margin" of the entire system. We now investigate the reactive power balancing conditions for such optimal reactive power allocation strategies. Consider a PV inverter network of which the communication network is defined by a directed graph $\mathcal{G} = (\mathcal{I}, \mathcal{A})$. The optimization problem is represented by Equation (4).

$$\min \ -s_T = -\sum_{i=1}^{N} \left[C_i - \frac{\sqrt{P_i^2 + x_i^2}}{3|V|} \right]$$

$$\text{s.t. } h(x) = \sum_{i=1}^{N} x_i - Q_D = 0 \tag{4}$$

$$g_i(x_i) = x_i - \sqrt{9|V|^2 C_i^2 - P_i^2} \leqslant 0, \ i = 1, \dots, N$$

$$g_{i+N}(x_i) = -x_i - \sqrt{9|V|^2 C_i^2 - P_i^2} \leqslant 0, \ i = 1, \dots, N$$

The optimal solutions of Equation (4) are represented as follows (the derivation of the optimal solutions is beyond the scope of this paper):

- If $\max_{i \in \mathcal{I}} \left\{ \dfrac{-\sqrt{9|V|^2 C_i^2 - P_i^2}}{P_i} \sum_{i \in \mathcal{I}} P_i \right\} \leqslant Q_D \leqslant \min_{i \in \mathcal{I}} \left\{ \dfrac{\sqrt{9|V|^2 C_i^2 - P_i^2}}{P_i} \sum_{i \in \mathcal{I}} P_i \right\}$, then for all $i \in \mathcal{I}$, the optimal reactive power is

$$x_i^* = \frac{P_i}{\sum_{i \in \mathcal{I}} P_i} Q_D, \ \forall \ i \in \mathcal{I} \tag{5}$$

- If $Q_D > \min_{i \in \mathcal{I}} \left\{ \dfrac{\sqrt{9|V|^2 C_i^2 - P_i^2}}{P_i} \sum_{i \in \mathcal{I}} P_i \right\} > 0$, then for all $i \in \mathcal{I}$, the optimal reactive power is

$$x_i^* = q_i^{\max}, \ i = 1, \dots, r$$

$$x_i^* = \frac{P_i}{\sum_{i=r+1}^{N} P_i} \left[Q_D - \sum_{i=1}^{r} q_i^{\max} \right], \ i = r+1, \dots, N \tag{6}$$

where we assume that all inverters are sorted in a sequence such that $\frac{q_1^{\max}}{P_1} \leqslant \frac{q_2^{\max}}{P_2} \leqslant \dots \leqslant \frac{q_N^{\max}}{P_N}$, and the number r, which is the number of inverters with saturated reactive power, is given by

$$r = \arg\min \left\{ r : \ \frac{P_i}{\sum_{i=r+1}^{N} P_i} \left[Q_D - \sum_{i=1}^{r} q_i^{\max} \right] < q_i^{\max}, \ i = r+1, \dots, N \right\} \tag{7}$$

- If $Q_D < \max_{i \in \mathcal{I}} \left\{ \dfrac{-\sqrt{9|V|^2 C_i^2 - P_i^2}}{P_i} \sum_{i \in \mathcal{I}} P_i \right\} < 0$, then for all $i \in \mathcal{I}$, the optimal reactive power is

$$x_i^* = q_i^{\min}, \ i = 1, \dots, t$$

$$x_i^* = \frac{P_i}{\sum_{i=t+1}^{N} P_i} \left[Q_D - \sum_{i=1}^{t} q_i^{\min} \right], \ i = t+1, \dots, N \tag{8}$$

where we assume that all inverters are sorted in a sequence such that $\frac{q_1^{\min}}{P_1} \geqslant \frac{q_2^{\min}}{P_2} \geqslant \dots \geqslant \frac{q_N^{\min}}{P_N}$, and the number t, which is the number of inverters with saturated reactive power, is given by

$$t = \arg\min \left\{ t : \ \frac{P_i}{\sum_{i=t+1}^{N} P_i} \left[Q_D - \sum_{i=1}^{t} q_i^{\min} \right] > q_i^{\min}, \ i = t+1, \dots, N \right\} \tag{9}$$

We still focus on the same simplex $\Delta = \{ x \in \mathbb{R}^N : \sum_{i=1}^{N} x_i = Q_D \}$ and assume Q_D is constant and known. In order to develop a class of passing strategies for the optimally allocated reactive power, we rewrite Equation (5) as

$$\frac{x_i^*}{P_i} = \frac{Q_D}{\sum_{i \in \mathcal{I}} P_i} \tag{10}$$

Equation (10) implies that the ratio of optimally allocated reactive power to the active power of all inverters with unsaturated reactive power is equal to the ratio of total reactive power to the total active power. Hence, the reactive power passing conditions are now modified based on $\frac{x_i}{P_i}$ instead of the reactive power x_i.

3.2.1. Reactive power passing strategies. In order to achieve a maximum "safety margin" of the system, the reactive power passing strategies are based on the equalization of the ratio of reactive power to the active power for each inverter. Also, because of the different capabilities of different inverters, it is possible to have some inverters with saturated reactive power at the bounds in the system. By taking these factors into account and assuming that the reactive power being passed between inverter is capacitive (negative) when $Q_D > 0$ (inductive when $Q_D < 0$), the following conditions define a class of optimally allocated reactive power passing strategies for inverter i at time k with the considerations of inverter bounds. When $Q_D > 0$, we assume that inverters can only pass capacitive reactive power between each other.

(b1) $\alpha_i^{i \to j} = 0$, if $\frac{1}{P_i(k)} x_i(k) - \frac{1}{P_j(k)} x_j^i(k) \geqslant 0$ or if $x_i(k) = q_i^{\max}(k)$;

(b2) $\frac{1}{P_i(k)} \left[x_i(k) - \sum_{\{j: j \in \mathcal{N}_i\}} \alpha_i^{i \to j} \right] \leqslant \min \left\{ \frac{1}{P_j(k)} \left[x_j^i(k) + \alpha_i^{i \to j} \right], \frac{1}{P_i(k)} \left[q_i^{\max}(k) + \alpha_i^{i \to j} \right] \right\}$,

$\forall \, j \in \mathcal{N}_i$ such that $\frac{1}{P_i(k)} x_i(k) - \frac{1}{P_j(k)} x_j^i(k) < 0$;

(b3) If $\alpha_i^{i \to j} < 0$ for some j, then $\alpha_i^{i \to j*} \leqslant \gamma_{ij*} \max \left\{ \frac{2[P_{j*}(k) x_i(k) - P_i(k) x_{j*}^i(k)]}{P_i(k) + P_{j*}(k)}, \left[x_i(k) - q_i^{\max} \right] \right\}$

where $j* = \arg \max_{j'} \left\{ \frac{x_{j'}^i(k)}{P_{j'}(k)} : j' \in \mathcal{N}_i \right\}$;

The conditions of $Q_D < 0$ are symmetric to the conditions of $Q_D > 0$, which are not presented here. Condition (b1) indicates that inverter i will not pass any capacitive reactive power to its neighboring inverter j if its reactive power perception about inverter j is optimally greater than its own reactive power, that is, if the ratio of reactive power to the active power of inverter i is greater than the corresponding ratio of inverter j, inverter i will not increase the reactive power level of itself to decrease the reactive power level of inverter j. Also, inverter i will not pass any capacitive reactive power to its neighboring inverters if the reactive power of inverter i reaches the upper bound, that is, inverter i cannot take more inductive reactive power for this case. Condition (b2) limits the amount of capacitive reactive power that inverter i can pass to its neighbor nodes then limits the increase of the reactive power level of inverter i. It indicates that after the reactive power transfer the ratio of reactive power to active power of inverter i must not be higher than the corresponding ratio of any of its neighbor inverters or the ratio of reactive power upper bound to active power of itself. This condition excludes the oscillation of reactive power between inverters. Condition (b3) implies that if inverter i is not optimally balanced with all of its neighbors, then it must pass some non-negligible amount of capacitive reactive power to the neighboring inverter with maximum optimal reactive power level. Meanwhile, the reactive power of inverter i is guaranteed not to exceed the upper bound. Condition (b3) is derived from

$$\frac{1}{2} \left[\frac{1}{P_i(k)} \alpha_i^{i \to j*} + \frac{1}{P_{j*}(k)} \alpha_i^{i \to j*} \right] \leqslant \gamma_{ij*} \left[\frac{1}{P_i(k)} x_i(k) - \frac{1}{P_{j*}(k)} x_{j*}(k) \right] \qquad (11)$$

where $j* = \arg \min_{j'} \left\{ \frac{x_{j'}^i(k)}{P_{j'}(k)} : j' \in \mathcal{N}_i \right\}$. Equation (11) directly implies that

$$\alpha_i^{i \to j*} \leqslant 2\gamma_{ij*} \frac{P_{j*}(k) x_i(k) - P_i(k) x_{j*}^i(k)}{P_i(k) + P_{j*}(k)} \qquad (12)$$

3.2.2. Distribution of optimal reactive power. The state equation of x_i with the reactive power passing conditions (b1)–(b3) is same as the one given by Equation (2). Let $\mathcal{X} = \Delta$ be the set of states and $x(k) = [x_1(k), \ldots, x_N(k)]^\top \in \mathcal{X}$ be the state vector, with $x_i(k)$ the reactive power of inverter i at time $k \geqslant 0$. Then, the set

$$
\begin{aligned}
\mathcal{X}_d = \Big\{ & x \in \mathcal{X} : \text{for all } i \in \mathcal{I}, \text{ either } \frac{x_i(k)}{P_i(k)} = \frac{x_j(k)}{P_j(k)} \text{ for all } (i,j) \in \mathcal{A} \text{ such that} \\
& q_i^{\min}(k) < x_j(k) < q_i^{\max}(k), \; \frac{x_i(k)}{P_i(k)} > \frac{x_j(k)}{P_j(k)} \text{ for all } (i,j) \in \mathcal{A} \text{ such that} \\
& x_j(k) = q_j^{\max}(k), \text{ and } \frac{x_i(k)}{P_i(k)} < \frac{x_j(k)}{P_j(k)} \text{ for all } (i,j) \in \mathcal{A} \text{ such that} \\
& x_j(k) = q_j^{\min}(k); \text{ or } x_i(k) = q_i^{\max}(k) \text{ or } q_i^{\min}(k) \Big\}
\end{aligned}
\tag{13}
$$

represents a distribution of the reactive power on the inverter network. Any distribution $x \in \mathcal{X}_d$ is such that for any $i \in \mathcal{I}$ either $x_i = q^{\max}(k)$ when $Q_D > 0$, $x_i = q^{\min}(k)$ when $Q_D < 0$; or if $q_i^{\min}(k) < x_i < q_i^{\max}(k)$, it must be the case that all neighboring inverters $j \in \mathcal{N}_i$ such that $q_j^{\min}(k) < x_j < q_j^{\max}(k)$ have the same ratio of reactive power to active power as inverter i. In $\mathcal{X}_c$, if $x_j = q_j^{\max}(k)$ when $Q_D > 0$ for $j \in \mathcal{N}_i$, then $\frac{1}{P_j} x_j \leqslant \frac{1}{P_i} x_i$; if $x_j = q_j^{\min}(k)$ when $Q_D < 0$ for $j \in \mathcal{N}_i$, then $\frac{1}{P_j} x_j \geqslant \frac{1}{P_i} x_i$. Notice that when $x(k) \in \mathcal{X}_d$, there is only one reactive power passing strategy that satisfies conditions (b1)–(b3), that is, $\alpha_j^{i \to j} = 0$ for all $i \in \mathcal{I}$. Similar to the uniformly distributed reactive power case, for any $x \in \mathcal{X}_d$, according to the definition of $\mathcal{X}_d$, it is possible that inverters in the subset $\mathcal{U}(k)$ are isolated (by the inverters with saturated reactive power) and have different optimal reactive power levels, that is, the ratio of reactive power to active power. Hence, there could be isolated "islands" of inverters in the network. The formation of inverter islands depends on the total reactive power, their initial distribution, the active power of each inverter, and the constraints on reactive power of each inverter, that is, $q_i^{\max}$ and $q_i^{\min}$.

Let us consider the reactive power distribution defined by Equation (13). The invariant set consists of many elements that represent different optimal reactive power distributions, and some distributions can lead to saturated reactive power on certain inverters (i.e., reactive power of that inverter hits the bounds). The next lemma shows that under certain situations, there is no inverter with saturated reactive power, and the distribution represented by the invariant set is unique.

Lemma 3 (Optimal distribution, unsaturated reactive power, uniqueness of invariant set)
If P_i, $q_i^{\max}$ and $q_i^{\min}$ are consistent with time k for all i and the total amount reactive power satisfies $\max_i \left\{ \frac{q_i^{\min}}{P_i} \right\} < \frac{Q_D}{\sum_{i=1}^{N} P_i} < \min_i \left\{ \frac{q_i^{\max}}{P_i} \right\}$, then the invariant set $\mathcal{X}_d$ satisfies $|\mathcal{X}_d| = 1$, and the invariant set $\mathcal{X}_d$ is simplified to $\mathcal{X}_d = \{ x \in \mathcal{X} : \text{for all } i \in \mathcal{I}, \frac{1}{P_i} x_i(k) = \frac{1}{P_j} x_j(k) \text{ for all } (i,j) \in \mathcal{A} \}$.

Proof
See Appendix A. □

Lemma 3 implies the conditions under which there is no inverter with saturated reactive power (i.e., no inverter's reactive power hits the bounds) and the uniqueness of the invariant set. All inverters will eventually have the same ratio of reactive power to active power, and the equalized ratio of reactive power to active power only depends on the total active power $\sum_{i=1}^{N} P_i$ and the total desired reactive power Q_D.

Next, let us assume a complete graph topology (i.e., every inverter connects to every other inverter). By adding assumption, we can loose the assumption on Q_D, then we have the following theorem:

Lemma 4 (Optimal distribution, complete graph, uniqueness of invariant set)
For a fully connected graph $(\mathcal{I}, \mathcal{A})$ and any total amount of reactive power such that $\sum_{i=1}^{N} q_i^{\min} < Q_D < \sum_{i=1}^{N} q_i^{\max}$, the invariant set $\mathcal{X}_d$ satisfies $|\mathcal{X}_d| = 1$.

Proof
See Appendix A. $\square$

Lemma 4 implies that for a fully connected graph topology, there is no isolated inverters with different reactive power to active power ratios. The full connectivity of the inverters leads to reactive power to active power ratio equalization across all inverters with unsaturated reactive power and the emergence (in some cases) of a set of inverters with saturated reactive power. Lemmas 3 and 4 studied the characteristics of the invariant set $\mathcal{X}_d$ that represents the optimal reactive power distribution for different total reactive power amount and connectivity topologies of the network. We now focus on the analysis of inverters approaching this set.

3.2.3. Stability analysis. Let us now consider again a general graph topology $(\mathcal{I}, \mathcal{A})$ and assume that every inverter is connected to the graph, but not every inverter connects to every other inverter. Also, we assume that the environmental conditions of the system are consistent with time k, that is, P_i, $q_i^{\max}$ and $q_i^{\min}$ are time-invariant.

Theorem 3 (Optimal distribution, asymptotic stability in large)
Given $(\mathcal{I}, \mathcal{A})$ and the reactive power passing strategies (b1)–(b3), there exists a constant Q_D such that the total desired reactive power Q_D satisfies $\max_i \left\{ \frac{q_i^{\min}}{P_i} \right\} < \frac{Q_D}{\sum_{i=1}^{N} P_i} < \min_i \left\{ \frac{q_i^{\max}}{P_i} \right\}$, then the invariant set $\mathcal{X}_d$ is asymptotically stable in large.

Proof
See Appendix A Because $\mathcal{X}_d$ is asymptotically stable in large, there is only one equilibrium distribution for each total amount reactive power Q_D which satisfies $\max_i \left\{ \frac{q_i^{\min}}{P_i} \right\} < \frac{Q_D}{\sum_{i=1}^{N} P_i} < \min_i \left\{ \frac{q_i^{\max}}{P_i} \right\}$. Thus, for any initial reactive power distribution, this equilibrium can be achieved. $\square$

We now assume more restrictive reactive power passing conditions in order to study the rate of convergence to the desired distribution. In particular, we assume

Assumption 3 (Rate of occurrence)
Every B time steps, there is the occurrence of the reactive power passing behaviors that are defined by conditions (b1)–(b3) for every inverter.

Then, the following theorem is derived:

Theorem 4 (Optimal distribution, exponential stability)
Given $(\mathcal{I}, \mathcal{A})$ and the reactive power passing strategies (b1)–(b3), there exists a constant Q_D such that the total desired reactive power Q_D satisfies $\max_i \left\{ \frac{q_i^{\min}}{P_i} \right\} < \frac{Q_D}{\sum_{i=1}^{N} P_i} < \min_i \left\{ \frac{q_i^{\max}}{P_i} \right\}$, then with Assumption 3, the invariant set $\mathcal{X}_d$ is exponentially stable in large.

Proof
See Appendix A. $\square$

It is shown in Theorems 3 and 4 the stability characteristics of the optimal reactive power distribution $\mathcal{X}_d$ with assumptions on the total amount of reactive power of the system. We now consider the stability of $\mathcal{X}_d$ for a more general Q_D but with the assumption of a complete graph.

Theorem 5 (Optimal distribution, complete graph, emergence of saturated inverters, asymptotic stability in large)
For a fully connected graph $(\mathcal{I}, \mathcal{A})$, any total amount of reactive power that satisfies $\sum_{i=1}^{N} q_i^{\min} < Q_D < \sum_{i=1}^{N} q_i^{\max}$, and the reactive power passing strategies (b1)–(b3), the invariant set $\mathcal{X}_d$ is asymptotically stable in large.

Proof
See A. Because we do not have the same restriction on Q_D as the one in Theorem 3, there can be some inverters with saturated reactive power in the network. However, Lemma 4 indicates the uniqueness of $\mathcal{X}_d$ for a fully connected graph. Then, any initial reactive power distribution will eventually converge to the unique equilibrium $\mathcal{X}_d$. The rate of the convergence to $\mathcal{X}_d$ with Assumption 3 for this case is given by the following theorem: □

Theorem 6 (Optimal distribution, complete graph, emergence of saturated inverters, exponential stability)
For a fully connected graph $(\mathcal{I}, \mathcal{A})$, any total amount of reactive power that satisfies $\sum_{i=1}^{N} q_i^{\min} < Q_D < \sum_{i=1}^{N} q_i^{\max}$, and the reactive power passing strategies (b1)–(b3), with Assumption 3 the invariant set $\mathcal{X}_d$ is exponentially stable in large.

Proof
See Appendix A.   □

4. SIMULATION: A CASE STUDY

Now, let us aim at a sample 1.5 MW grid-connected PV system with PV inverter network consisting of 8 PV inverters. These PV inverters are two different types of inverters. The required data of these inverters for the case study is shown in Table I. In order to distinguish each inverter from the others, we index these inverters from 1 to 8. Specifically, we index all five type 1 inverters to be inverters 1, 2, 4, 6, and 7; index all three type 2 inverters to be inverter 3, 5, and 8. Hence, $C_i = 301$ A for $i = 1, 2, 4, 6, 7$ and $C_i = 121$ A for $i = 3, 5, 8$. The nominal output voltage magnitude is 480 V AC, line to line. Then, $|V| = 480/\sqrt{3} = 277.1$ V. Here, we only investigate the ring topology shown in Figure 2 for the case that the reactive power is optimally distributed among all of the inverters for a maximized "safety margin". The optimal solutions indicate an equalized ratio between reactive power and active power. So we will focus on the ratio instead of the value of reactive power. For the optimal reactive power distribution, we consider a case that likely occurs for large-scale PV systems: the *partially shaded conditions*. We assume that the solar panels of all type 1 inverters are partially shaded by heavy clouds such that they have 0.1 solar irradiation. Also, we assume that the solar panels of inverter 5 (which is type 2 inverter) has a 0.1 solar irradiation profile as well. Inverters 3 and 8 have the same 0.9 solar profile. The output active power of each inverter is

$$
\begin{aligned}
P_i &= 0.1 P_i^{\max} = 0.1 \times 250 = 25 \text{ kW, for } i = 1, 2, 4, 6, 7 \\
P_i &= 0.1 P_i^{\max} = 0.1 \times 100 = 10 \text{ kW, for } i = 5 \\
P_i &= 0.9 P_i^{\max} = 0.9 \times 100 = 90 \text{ kW, for } i = 3, 8
\end{aligned}
\tag{14}
$$

Based on the active power given in Equation (14), the limits of reactive power of each inverter are

$$
q_i^{\min} = -\sqrt{9|V|^2 C_i^2 - P_i^2} = -248.99 \text{ kVar}, \ q_i^{\max} = -q_i^{\min} = 248.99 \text{ kVar}, \ i = 1, 2, 4, 6, 7
$$

$$
q_i^{\min} = -\sqrt{9|V|^2 C_i^2 - P_i^2} = -100.1 \text{ kVar}, \ q_i^{\max} = -q_i^{\min} = 100.1 \text{ kVar}, \ i = 5
$$

$$
q_i^{\min} = -\sqrt{9|V|^2 C_i^2 - P_i^2} = -44.94 \text{ kVar}, \ q_i^{\max} = -q_i^{\min} = 44.94 \text{ kVar}, \ i = 3, 8
\tag{15}
$$

Table I. Data of the inverters in the sample grid-connected PV system.

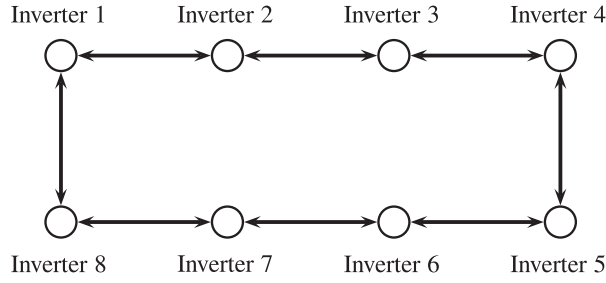|	Type 1 inverter	Type 2 inverter
Maximum output power	250 kW	100 kW
Nominal output voltage	480 V	480 V (AC, line to line)
Nominal output current	301 A	121 A
Nominal output frequency	60 Hz	60 Hz
Number of inverters	5	3

Figure 2. The ring topology of the communication system of the DC–AC inverter network.
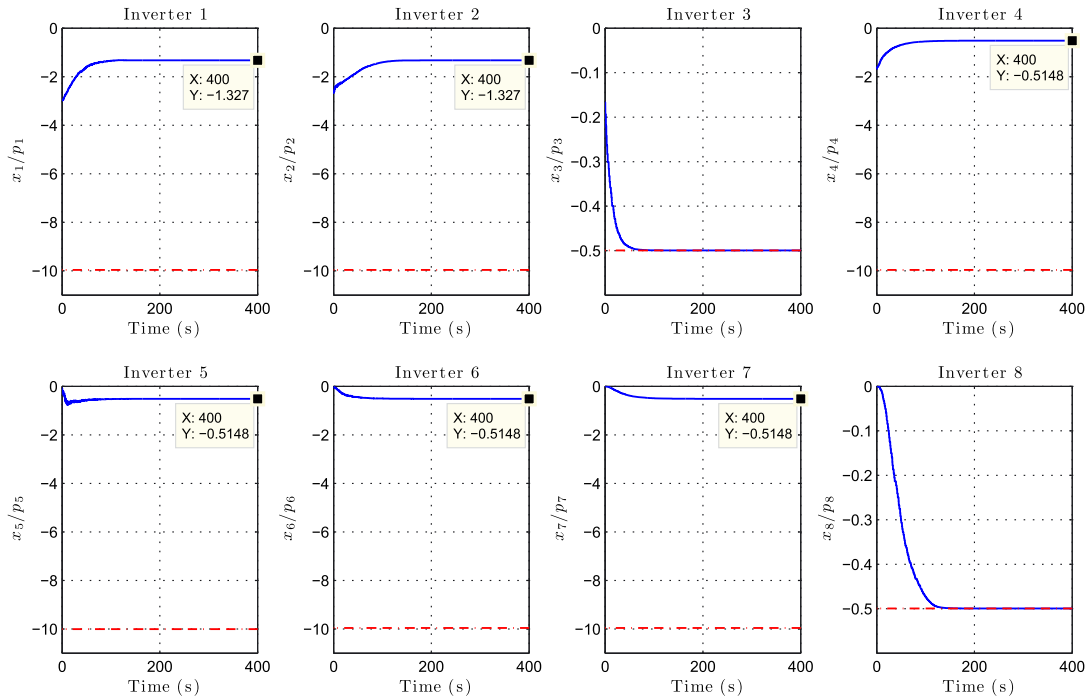

Figure 3. The ratio of optimally distributed reactive power to active power with saturated inverters for a ring connection topology. The solid line of each subplot: the ratio of reactive power to active power; the dashed line of each subplot: the ratio of reactive power lower bound to active power.

Then, the ratio of reactive power lower bound to active power and the ratio of reactive power upper bound to active power for each inverter are

$$\frac{q_i^{\min}}{P_i} = \frac{-248.99}{25} = -9.9596, \ \frac{q_i^{\max}}{P_i} = -\frac{q_i^{\min}}{P_i} = 9.9596, \ \text{for } i = 1,2,4,6,7$$

$$\frac{q_i^{\min}}{P_i} = \frac{-100.1}{10} = -10.01, \ \frac{q_i^{\max}}{P_i} = -\frac{q_i^{\min}}{P_i} = 10.01, \ \text{for } i = 5 \tag{16}$$

$$\frac{q_i^{\min}}{P_i} = \frac{-44.94}{25} = -0.4993, \ \frac{q_i^{\max}}{P_i} = -\frac{q_i^{\min}}{P_i} = 0.4993, \text{for } i = 3,8$$

The total desired reactive power is still $Q_D = -200\,\text{kVar}$ for this case, and the ratio of total reactive power to total active power is
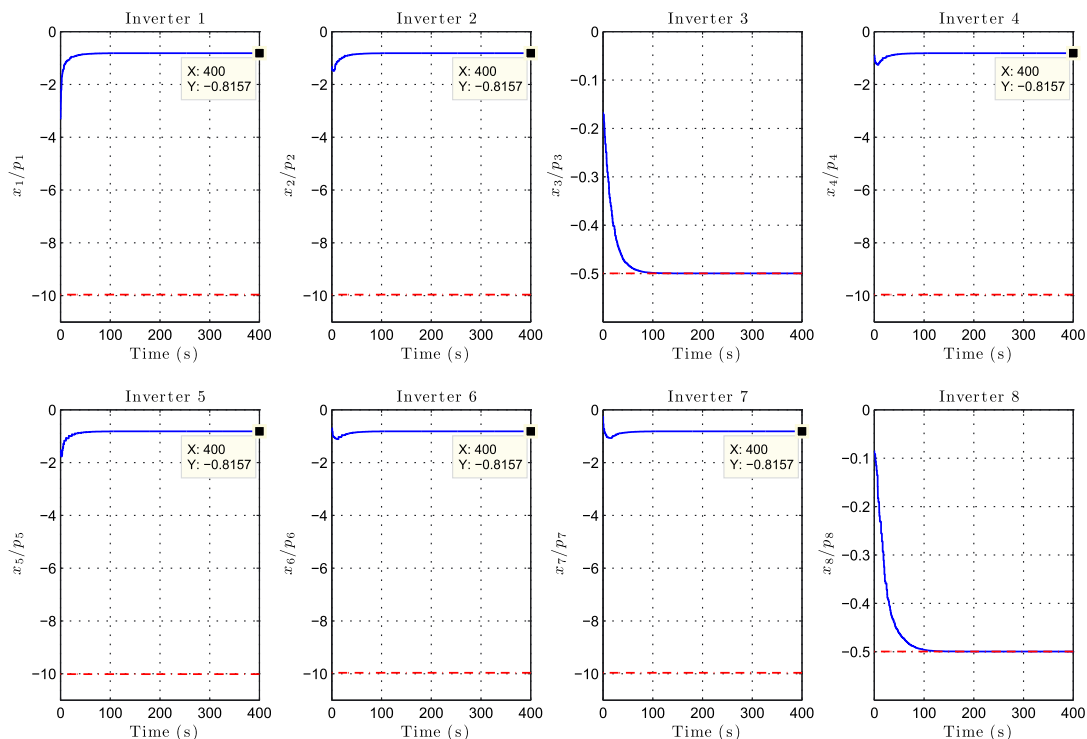
Figure 4. The ratio of optimally distributed reactive power to active power with saturated inverters for a fully connected graph. The solid line of each subplot: the ratio of reactive power to active power; the dashed line of each subplot: the ratio of reactive power lower bound to active power.

$$\frac{Q_D}{\sum_{i=1}^{N} P_i} = \frac{-200}{25 \times 5 + 90 \times 2 + 10} = -\frac{200}{315} = -0.6349 \tag{17}$$

Hence, Lemma 3 is not satisfied for this case, and there are saturated inverters in the system (which are inverters 3 and 8). Figure 3 shows the reactive power balancing (to have an equalized ratio) for partially shaded conditions with a ring topology of the system. It indicates that due to the saturated inverters 3 and 8, inverters 1 and 2 have an equal ratio of the reactive power to active power while inverters 4–7 have a different equalized ratio. This is because saturated inverters 3 and 8 isolate them to form two islands. In order to avoid the emergence of inverter islands, a fully connected graph is used. Figure 4 shows that the ratio of reactive power to active power of all unsaturated inverters are equal for a fully connected graph.

5. CONCLUSIONS

In this paper, a distributed reactive power control based on balancing strategies is proposed for the grid-connected PV inverter network. Reactive power balancing strategies are designed for uniform reactive power distribution and optimal reactive power distribution. Invariant sets are defined to denote the desired reactive power distributions. By using the proposed reactive power balancing strategies, the invariant sets are mathematically proved to be asymptotically stable and exponentially stable under certain assumptions. Simulation results are derived from a case study for both reactive power distributions by considering different initial conditions to validate the reactive power balancing control. Because the control strategies proposed in this paper is generic without considering specific systems where the grid-connected PV inverter network works, one possible future research direction is the distributed control development for PV inverter network in certain systems such as the distribution systems.

APPENDIX A: PROOFS OF THEOREMS

Proof of Lemma 1

If $q_i^{\max}$ and $q_i^{\min}$ are consistent with time k, and if $N \max_i\{q_i^{\min}\} < Q_D < N \min_i\{q_i^{\max}\}$, for any $x \in \mathcal{X}_c$, we have $x_i = Q_D/N$, which implies that $q_i^{\min} < x_i < q_i^{\max}$ for all $i \in \mathcal{I}$, that is, the reactive power of all inverters will not saturate at the bounds. Because we assume Q_D is constant, there is only one reactive power level for all inverters, that is, Q_D/N. Hence, we conclude that $|\mathcal{X}_c| = 1$.

Proof of Lemma 2

With fixed $q_i^{\max}$ and $q_i^{\min}$ for all $i \in \mathcal{I}$ (as indicated by condition (ii) of Assumption 1), if $N \max_i\{q_i^{\min}\} < Q_D < N \min_i\{q_i^{\max}\}$, this leads to the case of 1; if $N \min_i\{q_i^{\max}\} < Q_D < \sum_{i=1}^N q_i^{\max}$ for $Q_D > 0$, or $\sum_{i=1}^N q_i^{\min} < Q_D < N \max_i\{q_i^{\min}\}$ for $Q_D < 0$, the reactive power of some inverters saturate at the bounds. If in addition we assume a complete graph, that is, the case given by condition (i) of Assumption 1, there are no "isolated" inverters because of some saturated inverters. The unsaturated inverters are connected together and have the same reactive power level. Moreover, if we assume there are $r < N$ number of saturated inverters (this number depends on Q_D, $q_i^{\max}$, and $q_i^{\max}$), then we know that the inverters with saturated reactive power are the first r inverters in the sequence such that $q_1^{\max} \leqslant q_2^{\max} \leqslant \ldots \leqslant q_N^{\max}$ for $Q_D > 0$ and $q_1^{\min} \geqslant q_2^{\min} \geqslant \ldots \geqslant q_N^{\min}$ for $Q_D < 0$. The unsaturated inverters have the same reactive power level $(Q_D - \sum_{i=1}^r x_i)/(N - r)$.

Proof of Theorem 1

Recall that $\mathcal{U}$ is the subset of inverters with unsaturated reactive power and $\mathcal{S}$ is the subset of inverters with saturated reactive power. The terms $|\mathcal{U}|$ and $|\mathcal{S}|$ denote the numbers of elements in $\mathcal{U}$ and $\mathcal{S}$, that is, the number of inverters with unsaturated reactive power and the number of inverters with saturated reactive power, respectively.

First, let us consider the case that $Q_D > 0$. With Assumption 1, the invariant set becomes

$$\mathcal{X}_c^+ = \left\{x \in \mathcal{X} : \text{for all } i \in \mathcal{U}, x_i(k) = x_j(k), \text{ for all } j \in \mathcal{U}; x_i(k) = q_i^{\max}(k) \text{ for all } i \in \mathcal{S}\right\} \tag{18}$$

Consider the state $\bar{x} \in \mathcal{X}_c^+$, define $\mathcal{S}_c$ to be the subset of inverters such that for all $i \in \mathcal{S}_c$, $\bar{x}_i = q_i^{\max}$ and define $\mathcal{U}_c$ to be the subset of inverters such that for all $i \in \mathcal{U}_c$, $\bar{x}_i < q_i^{\max}$. As discussed previously, we know that for any $\bar{x} \in \mathcal{X}_c^+$,

$$\begin{aligned} \bar{x}_i &= q_i^{\max}, \text{ for all } i \in \mathcal{S}_c \\ \bar{x}_i &= \frac{1}{|\mathcal{U}_c|}\left[Q_D - \sum_{j \in \mathcal{S}_c} x_j\right], \text{ for all } i \in \mathcal{U}_c \end{aligned} \tag{19}$$

and

$$q_i^{\max} \leqslant \frac{1}{|\mathcal{U}_c|}\left[Q_D - \sum_{j \in \mathcal{S}_c} x_j\right], \text{ for all } i \in \mathcal{S}_c \tag{20}$$

Choose

$$\rho(x(k), \mathcal{X}_c^+) = \inf\left\{\max_{i \in \mathcal{I}}\{|x_i(k) - \bar{x}_i|\} : \bar{x} \in \mathcal{X}_c^+\right\} \tag{21}$$

and

$$V(x(k)) = \max_{i \in \mathcal{U}_c}\left\{\frac{1}{|\mathcal{U}_c|}\sum_{j \in \mathcal{U}_c} x_j(k) - x_i(k)\right\} + \sum_{i \in \mathcal{S}_c}|x_i(k) - q_i^{\max}| \tag{22}$$

From Equation (19), we know that

$$\rho\left(x(k), \mathcal{X}_c^+\right) \geqslant \frac{1}{2}\left[\max_{i \in \mathcal{U}_c}\{x_i(k)\} - \min_{i \in \mathcal{U}_c}\{x_i(k)\}\right] \tag{23}$$

and

$$\rho\left(x(k), \mathcal{X}_c^+\right) \leqslant \left[\max_{i \in \mathcal{U}_c}\{x_i(k)\} - \min_{i \in \mathcal{U}_c}\{x_i(k)\}\right] + \sum_{i \in \mathcal{S}_c}|x_i(k) - q_i^{\max}| \tag{24}$$

The reason that Equation (24) holds is as follows:

- At time k, if $\min_{i \in \mathcal{U}_c}\{x_i(k)\} \leqslant \bar{x}_i$ for $i \in \mathcal{U}_c$, then $\max_{i \in \mathcal{U}_c}\{x_i(k)\} - \min_{i \in \mathcal{U}_c}\{x_i(k)\} \geqslant \max_{i \in \mathcal{U}_c}\{|x_i(k) - \bar{x}_i|\}$. It is obvious that Equation (24) holds;
- At time k, if $\min_{i \in \mathcal{U}_c}\{x_i(k)\} \geqslant \bar{x}_i$ for $i \in \mathcal{U}_c$, then $\max_{i \in \mathcal{U}_c}\{x_i(k)\} - \min_{i \in \mathcal{U}_c}\{x_i(k)\} \leqslant \max_{i \in \mathcal{U}_c}\{|x_i(k) - \bar{x}_i|\}$. However, $\max_{i \in \mathcal{U}_c}\{x_i(k)\} - \min_{i \in \mathcal{U}_c}\{x_i(k)\} + \sum_{i \in \mathcal{S}_c}|x_i(k) - q_i^{\max}| \geqslant \max_{i \in \mathcal{U}_c}\{|x_i(k) - \bar{x}_i|\}$, because for this case $\max_{i \in \mathcal{U}_c}\{|x_i(k) - \bar{x}_i|\} \leqslant \sum_{i \in \mathcal{S}_c}|x_i(k) - q_i^{\max}|$ for $i \in \mathcal{U}_c$, that is, the maximum difference between $x_i(k)$ and the final equalized reactive power level of inverter $i \in \mathcal{U}_c$ is less than the total difference between current reactive power levels and the final saturated reactive power levels of inverters in the subset $\mathcal{S}_c$. In other words, because $\min_{i \in \mathcal{U}_c}\{x_i(k)\} \geqslant \bar{x}_i$ for $i \in \mathcal{U}_c$, all inverters in the subset of $\mathcal{U}_c$ will decrease their reactive power levels to the final equalized level by passing reactive power to inverters in the subset of $\mathcal{S}_c$ with the reactive power passing strategies (a1)–(a3). Hence, $\max_{i \in \mathcal{U}_c}\{x_i(k)\} - \min_{i \in \mathcal{U}_c}\{x_i(k)\} + \sum_{i \in \mathcal{S}_c}|x_i(k) - q_i^{\max}| \geqslant \max_{i \in \mathcal{U}_c}\{|x_i(k) - \bar{x}_i|\}$ implies Equation (24).

Equation (22) implies that

$$\begin{aligned}
V(x(k)) &= \frac{1}{|\mathcal{U}_c|}\sum_{j \in \mathcal{U}_c} x_j(k) - \min_{i \in \mathcal{U}_c}\{x_i(k)\} + \sum_{i \in \mathcal{S}_c}|x_i(k) - q_i^{\max}| \\
&\leqslant \max_{i \in \mathcal{U}_c}\{x_i(k)\} - \min_{i \in \mathcal{U}_c}\{x_i(k)\} + \sum_{i \in \mathcal{S}_c}|x_i(k) - q_i^{\max}|
\end{aligned} \tag{25}$$

Equation (23) implies that

$$2\rho(x(k), \mathcal{X}_c^+) \geqslant \max_{i \in \mathcal{U}_c}\{x_i(k)\} - \min_{i \in \mathcal{U}_c}\{x_i(k)\}$$
$$2\rho(x(k), \mathcal{X}_c^+) + \sum_{i \in \mathcal{S}_c}|x_i(k) - q_i^{\max}| \geqslant \max_{i \in \mathcal{U}_c}\{x_i(k)\} - \min_{i \in \mathcal{U}_c}\{x_i(k)\} + \sum_{i \in \mathcal{S}_c}|x_i(k) - q_i^{\max}| \tag{26}$$

We also know that

$$\rho(x(k), \mathcal{X}_c^+) \geqslant \max_{i \in \mathcal{S}_c}\{|x_i(k) - q_i^{\max}|\}$$
$$|\mathcal{S}_c|\rho(x(k), \mathcal{X}_c) \geqslant |\mathcal{S}_c|\max_{i \in \mathcal{S}_c}\{|x_i(k) - q_i^{\max}|\} \geqslant \sum_{i \in \mathcal{S}_c}|x_i(k) - q_i^{\max}| \tag{27}$$

Hence, we obtain from Equations (25)–(27) that

$$\begin{aligned}
V(x(k)) &\leqslant \max_{i \in \mathcal{U}_c}\{x_i(k)\} - \min_{i \in \mathcal{U}_c}\{x_i(k)\} + \sum_{i \in \mathcal{S}_c}|x_i(k) - q_i^{\max}| \\
&\leqslant (2 + |\mathcal{S}_c|)\rho(x(k), \mathcal{X}_c^+)
\end{aligned} \tag{28}$$

Notice that

$$\frac{1}{|\mathcal{U}_c|}\sum_{j \in \mathcal{U}_c} x_j(k) \geqslant \frac{1}{|\mathcal{U}_c|}\left[\max_{i \in \mathcal{U}_c}\{x_i(k)\} + (|\mathcal{U}_c| - 1)\min_{i \in \mathcal{U}_c}\{x_i(k)\}\right] \tag{29}$$

Combining Equations (25) and (29), we obtain

$$
\begin{aligned}
V(x(k)) &\geqslant \frac{1}{|\mathcal{U}_c|}\left[\max_{i\in\mathcal{U}_c}\{x_i(k)\} + (|\mathcal{U}_c|-1)\min_{i\in\mathcal{U}_c}\{x_i(k)\}\right] - \min_{i\in\mathcal{U}_c}\{x_i(k)\} + \sum_{i\in\mathcal{S}_c}|x_i(k)-q_i^{\max}| \\
&= \frac{1}{|\mathcal{U}_c|}\left[\max_{i\in\mathcal{U}_c}\{x_i(k)\} - \min_{i\in\mathcal{U}_c}\{x_i(k)\}\right] + \sum_{i\in\mathcal{S}_c}|x_i(k)-q_i^{\max}| \\
&= \frac{1}{|\mathcal{U}_c|}\left[\max_{i\in\mathcal{U}_c}\{x_i(k)\} - \min_{i\in\mathcal{U}_c}\{x_i(k)\} + |\mathcal{U}_c|\sum_{i\in\mathcal{S}_c}|x_i(k)-q_i^{\max}|\right] \\
&\geqslant \frac{1}{|\mathcal{U}_c|}\rho\left(x(k),\mathcal{X}_c^+\right)
\end{aligned}
\tag{30}
$$

Hence, Equations (25) and (30) imply that

$$
\frac{1}{|\mathcal{U}_c|}\rho\left(x(k),\mathcal{X}_c^+\right) \leqslant V(x(k)) \leqslant (2+|\mathcal{S}_c|)\rho\left(x(k),\mathcal{X}_c^+\right)
\tag{31}
$$

Thus,

- For $c_1 = \frac{1}{2}\left[\max_{i\in\mathcal{U}_c}\{x_i(k)\} - \min_{i\in\mathcal{U}_c}\{x_i(k)\}\right] > 0$, it is possible to find a $c_2 = \frac{1}{|\mathcal{U}_c|}\left[\max_{i\in\mathcal{U}_c}\{x_i(k)\} - \min_{i\in\mathcal{U}_c}\{x_i(k)\}\right] > 0$ such that $V(x(k)) \geqslant c_2$ and $\rho(x(k),\mathcal{X}_c^+) \geqslant c_1$;

- For $c_3 = \left[\max_{i\in\mathcal{U}_c}\{x_i(k)\} - \min_{i\in\mathcal{U}_c}\{x_i(k)\}\right] + \sum_{i\in\mathcal{S}_c}\left|x_i(k)-q_i^{\max}\right| > 0$, it is possible to find a $c_4 = (2+|\mathcal{S}_c|)c_3$ such that when $\rho(x(k),\mathcal{X}_c^+) \leqslant c_3$, we have $V(x(k)) \leqslant c_4$;

- The function $V(x(k))$ is non-increasing with the reactive power passing strategies (a1)–(a4). The reasons are as follow:

  - At time k, if $\min_{i\in\mathcal{U}_c}\{x_i(k)\} \leqslant \bar{x}_i$ for $i\in\mathcal{U}_c$, then the average reactive power level $\frac{1}{|\mathcal{U}_c|}\sum_{j\in\mathcal{U}_c}x_j(k)$ tends to decrease to $\bar{x}_i$ because of the capacitive reactive power passed from the inverters in the subset of $\mathcal{S}_c$. Also, because $\min_{i\in\mathcal{U}_c}\{x_i(k)\} \leqslant \bar{x}_i$, the inverter with the reactive power of $\min_{i\in\mathcal{U}_c}\{x_i(k)\}$ tends to pass capacitive reactive power to others to increase its reactive power level that makes $\min_{i\in\mathcal{U}_c}\{x_i(k)\}$ increase and $-\min_{i\in\mathcal{U}_c}\{x_i(k)\}$ decrease, or $\min_{i\in\mathcal{U}_c}\{x_i(k)\}$ can decrease because of some capacitive reactive power it receives from inverters in the subset of $\mathcal{S}_c$. However, the reactive power increase of some inverters in the subset of $\mathcal{S}_c$ cancels the decrease of $\min_{i\in\mathcal{I}_c}\{x_i(k)\}$. Now consider the term $\sum_{i\in\mathcal{S}_c}\left|x_i(k)-q_i^{\max}\right|$. Because we assume the graph is complete, that is, each inverter (node) in the graph is fully connected to other inverters, and $\bar{x}_i \geqslant \bar{x}_j$ for $i\in\mathcal{I}_c$ and $j\in\mathcal{S}_c$, then $\sum_{i\in\mathcal{S}_c}\left|x_i(k)-q_i^{\max}\right|$ tends to decrease because of the capacitive reactive power that the inverters in the subset $\mathcal{S}_c$ passes to inverters in the subset of $\mathcal{U}_c$, that is, inverters in the subset $\mathcal{S}_c$ tends to increase their reactive power levels.

  - At time k, if $\min_{i\in\mathcal{U}_c}\{x_i(k)\} \geqslant \bar{x}_i$ for $i\in\mathcal{U}_c$, it is possible that $\min_{i\in\mathcal{U}_c}\{x_i(k)\}$ decreases and $-\min_{i\in\mathcal{U}_c}\{x_i(k)\}$ increases. However, for this case, all inverters in the subset of $\mathcal{U}_c$ receives capacitive power from inverters in the subset of $\mathcal{S}_c$, then the decrease of $\min_{i\in\mathcal{U}_c}\{x_i(k)\}$ is not greater than the decrease of $\sum_{i\in\mathcal{S}_c}\left|x_i(k)-q_i^{\max}\right|$. Hence, the function $V(x(k))$ is non-increasing.

- Furthermore, with the reactive power passing strategies defined by (a1)–(a3) $V(x(k)) \to 0$ as $k \to 0$ for all $x(k) \in \mathcal{X}$.

Then, we conclude that with the reactive power passing strategies defined by conditions (a1)–(a3), the invariant set $\mathcal{X}_c^+ = \{x \in \mathcal{X} : \text{for all } i \in \mathcal{U}, x_i(k) = x_j(k), \text{for all } j \in \mathcal{U}; x_i(k) = q_i^{\max}(k) \text{ for all } i \in \mathcal{S}\}$ is asymptotically stable in large. Similarly, we can also prove that the invariant set $\mathcal{X}_c^- = \{x \in \mathcal{X} : \text{for all } i \in \mathcal{U}, x_i(k) = x_j(k), \text{ for all } j \in \mathcal{U}; x_i(k) = q_i^{\min}(k) \text{ for all } i \in \mathcal{S}\}$ is asymptotically stable in large for the case that $Q_D < 0$. Hence, the invariant set $\mathcal{X}_c$ is asymptotically stable in large.

Proof of Theorem 2
First, let us consider the case that $Q_D > 0$. With Assumption 1, the invariant set becomes $\mathcal{X}_c^+$ that is given in Equation (18). We choose $\rho(x(k), \mathcal{X}_c^+)$ the same as the one in Equation (21) and the Lyapunov function

$$V(x(k)) = \max_{i \in \mathcal{U}_c}\left\{x_i(k) - \frac{1}{|\mathcal{U}_c|}\sum_{j \in \mathcal{U}_c} x_j(k)\right\} + \frac{1}{|\mathcal{U}_c|}\sum_{i \in \mathcal{S}_c} |x_i(k) - q_i^{\max}| \tag{32}$$

Equation (24) is rewritten as

$$\rho(x(k), \mathcal{X}_c^+) \leqslant \left[\max_{i \in \mathcal{I}}\{x_i(k)\} - \min_{i \in \mathcal{U}_c}\{x_i(k)\} + \max_{i \in \mathcal{S}_c}|x_i(k) - q_i^{\max}|\right] \tag{33}$$

Equation (33) holds because

- it is obvious that $\max\limits_{i \in \mathcal{S}_c}\left|x_i(k) - q_i^{\max}\right|$ is identical with $\max\limits_{j \in \mathcal{S}_c}\{|x_j(k) - \bar{x}_j|\}$.
- Also, it is obvious that $\min\limits_{i \in \mathcal{U}_c}\{x_i(k)\} <= \frac{1}{|\mathcal{U}_c|}\sum\limits_{j \in \mathcal{U}_c} x_j(k)$, so $\max\limits_{i \in \mathcal{I}}\{x_i(k)\} - \min\limits_{i \in \mathcal{U}_c}\{x_i(k)\} > \max\limits_{j \in \mathcal{U}_c}\{|x_j(k) - \bar{x}_j|\}$.

Hence, $\rho(x(k), \mathcal{X}_c^+) = \inf\left\{\max\limits_{i \in \mathcal{I}}\{|x_i(k) - \bar{x}_i|\} : \bar{x} \in \mathcal{X}_c^+\right\} \leqslant \left[\max\limits_{i \in \mathcal{I}}\{x_i(k)\} - \min\limits_{i \in \mathcal{I}}\{x_i(k)\}\right]$. Equation (32) implies that

$$\begin{aligned}V(x(k)) &= \max_{i \in \mathcal{U}_c}\{x_i(k)\} - \frac{1}{|\mathcal{U}_c|}\sum_{j \in \mathcal{U}_c} x_j(k) + \frac{1}{|\mathcal{U}_c|}\sum_{i \in \mathcal{S}_c} |x_i(k) - q_i^{\max}|\\ &\leqslant \max_{i \in \mathcal{U}_c}\{x_i(k)\} - \min_{i \in \mathcal{U}_c}\{x_i(k)\} + \sum_{i \in \mathcal{S}_c} |x_i(k) - q_i^{\max}|\end{aligned} \tag{34}$$

Hence, from Equations (26), (27), and (34), we arrive at the same result as Equation (28), that is, $V(x(k)) \leqslant (2 + |\mathcal{S}_c|)\rho(x(k), \mathcal{X}_c^+)$. Similar to Equation (30), we obtain

$$\begin{aligned}V(x(k)) &\geqslant \max_{i \in \mathcal{U}_c}\{x_i(k)\} - \frac{1}{|\mathcal{U}_c|}\left[(|\mathcal{U}_c| - 1)\max_{i \in \mathcal{U}_c}\{x_i(k)\} + \min_{i \in \mathcal{U}_c}\{x_i(k)\}\right] + \frac{1}{|\mathcal{U}_c|}\sum_{i \in \mathcal{S}_c} |x_i(k) - q_i^{\max}|\\ &= \frac{1}{|\mathcal{U}_c|}\left[\max_{i \in \mathcal{U}_c}\{x_i(k)\} - \min_{i \in \mathcal{U}_c}\{x_i(k)\}\right] + \frac{1}{|\mathcal{U}_c|}\sum_{i \in \mathcal{S}_c} |x_i(k) - q_i^{\max}|\\ &= \frac{1}{|\mathcal{U}_c|}\left[\max_{i \in \mathcal{U}_c}\{x_i(k)\} - \min_{i \in \mathcal{U}_c}\{x_i(k)\} + \sum_{i \in \mathcal{S}_c} |x_i(k) - q_i^{\max}|\right]\\ &\geqslant \frac{1}{|\mathcal{U}_c|}\rho\left(x(k), \mathcal{X}_c^+\right)\end{aligned} \tag{35}$$

Hence, we have

$$\frac{1}{|\mathcal{U}_c|}\rho\left(x(k), \mathcal{X}_c^+\right) \leqslant V(x(k)) \leqslant (2 + |\mathcal{S}_c|)\rho\left(x(k), \mathcal{X}_c^+\right) \tag{36}$$

Let $\gamma = \min_{i,j \in \mathcal{I}} \{\gamma_{ij}\}$. For any $i \in \mathcal{I}$ and $k \geqslant 0$, we know from condition (a2) that if the reactive power passing occurs for inverter i, and if $\alpha_i^{i \to j} < 0$, then $\alpha_i^{i \to j} \leqslant \gamma \left[x_i(k) - x_j^i(k) \right]$. We have $x_i(k+1) \leqslant x_j^i(k) + \gamma \left[x_i(k) - x_j^i(k) \right]$ for $j \in \mathcal{N}_i$. If the reactive power passing does not occur or $\alpha_i^{i \to j} = 0$, then $x_i(k+1) = x_i(k)$. It follows that in any case,

$$x_i(k+1) \leqslant \max_{i \in \mathcal{I}} \{x_i(k)\} + \gamma \left[x_i(k) - \max_{i \in \mathcal{I}} \{x_i(k)\} \right], \ \forall \, i \in \mathcal{I} \tag{37}$$

Because $\bar{x}_i > \max_{j \in \mathcal{S}_c} \{q_j^{\max}\}$ for $i \in \mathcal{U}_c$, $\max_{i \in \mathcal{U}_c} \{x_i(k)\} > \max_{j \in \mathcal{S}_c} \{q_j^{\max}\}$ always holds. Then, $\max_{i \in \mathcal{I}} \{x_i(k)\}$ is a non-increasing function of k. We now show via induction that

$$x_i(k+t) \leqslant \max_{i \in \mathcal{I}} \{x_i(k)\} + \gamma^t \left[x_i(k) - \max_{i \in \mathcal{I}} \{x_i(k)\} \right], \ \forall \, i \in \mathcal{I} \tag{38}$$

for all $t \geqslant 0$. When $t = 1$, Equation (38) is turned to be Equation (37). Now, we assume that Equation (38) holds for an arbitrary t and show that Equation (38) also holds for the case of $t + 1$. According to Equation (37) for any $i \in \mathcal{I}$ at time $k + t + 1$, we have

$$
\begin{aligned}
x_i(k+t+1) &\leqslant \max_{i \in \mathcal{I}} \{x_i(k+t)\} + \gamma \left[x_i(k+t) - \max_{i \in \mathcal{I}} \{x_i(k+t)\} \right] \\
&\leqslant \max_{i \in \mathcal{I}} \{x_i(k)\} + \gamma \left[x_i(k+t) - \max_{i \in \mathcal{I}} \{x_i(k)\} \right] \\
&\leqslant \max_{i \in \mathcal{I}} \{x_i(k)\} + \gamma \left[\max_{i \in \mathcal{I}} \{x_i(k)\} + \gamma^t \left[x_i(k) - \max_{i \in \mathcal{I}} \{x_i(k)\} \right] - \max_{i \in \mathcal{I}} \{x_i(k)\} \right] \\
&\leqslant \max_{i \in \mathcal{I}} \{x_i(k)\} + \gamma^{t+1} \left[x_i(k) - \max_{i \in \mathcal{I}} \{x_i(k)\} \right]
\end{aligned}
\tag{39}
$$

Thus, Equation (38) must be valid for all $t \geqslant 0$.

- Fix $i \in \mathcal{U}_c$ and $k \geqslant 0$, we now show that reactive power of all neighbors of i are bounded from above for all k', $k' \geqslant k + B$. Specifically, we will show that

$$x_j(k') \leqslant \max_{i \in \mathcal{I}} \{x_i(k)\} + \gamma^{k'-k} \left[x_i(k) - \max_{i \in \mathcal{I}} \{x_i(k)\} \right], \ \forall \, k' \geqslant k + B, j \in \mathcal{N}_i \tag{40}$$

Because we assume a fully connected graph, Equation (40) is turned into

$$\max_{i \in \mathcal{I}} \{x_i(k')\} \leqslant \max_{i \in \mathcal{I}} \{x_i(k)\} + \gamma^{k'-k} \left[x_i(k) - \max_{i \in \mathcal{I}} \{x_i(k)\} \right], \ \forall \, k' \geqslant k + B, i \in \mathcal{U}_c \tag{41}$$

There are times $k_p \geqslant k$, $p \in \{1, 2, \dots\}$ such that the reactive power passing occurs for inverter i, and the reactive power passing does not occur for $k' \neq k_p$. We know from Assumption 2 that $k \leqslant k_1 < k + B$, $k_{p-1} < k_p < k_{p-1} + B$, $\forall \, p \in \{2, 3, \dots\}$. Now let us consider two cases:

– Let us consider a time k_p, $p \in \{1, 2, \dots\}$, and $j \in \mathcal{N}_i$ such that $x_j(k_p) > x_i(k_p)$, that is, at time k_p, inverter i passes a non-negligible amount of capacitive reactive power to inverter j. According to condition (a2), we have

$$x_j(k_p) - \sum_r \alpha_r^{j \to r} \leqslant x_r(k_p) + \alpha_r^{j \to r}, \ \forall \, r \in \mathcal{N}_j \text{ such that } x_j(k_p) < x_r(k_p) \tag{42}$$

Equation (42) implies that

$$x_j(k_p) - \sum_r \alpha_r^{j \to r} \leqslant x_{r*}(k_p) + \alpha_{r*}^{j \to r*}, \text{ for some } r \in \{r : x_r \geqslant x_{r'}, \ \forall \, r' \in \mathcal{N}_j\} \tag{43}$$

From time k_p to $k_p + 1$, we have

$$x_j(k_p + 1) = x_j(k_p) - \sum_r \alpha_r^{j \to r} + \sum_{r'} \alpha_{r'}^{r' \to j},$$
$$\forall\, r, r' \in \mathcal{N}_j \text{ such that } x_j(k_p) < x_r(k_p) \text{ and } x_j(k_p) > x_{r'}(k_p) \tag{44}$$

As $i \in \mathcal{N}_j$, that is, inverter i is one of the neighboring inverter of inverter j, and $x_j(k_p) > x_i(k_p)$, Equation (44) becomes

$$x_j(k_p + 1) = x_j(k_p) - \sum_r \alpha_r^{j \to r} + \sum_{r',r' \neq i} \alpha_{r'}^{r' \to j} + \alpha_j^{i \to j} \tag{45}$$

Equations (43)–(45) imply that

$$x_j(k_p + 1) \leqslant x_{r^*}(k_p) + \alpha_{r^*}^{j \to r^*} + \sum_{r',r' \neq i} \alpha_{r'}^{r' \to j} + \alpha_j^{i \to j} \tag{46}$$

From condition (a3), we know that $\alpha_j^{i \to j} \leqslant \gamma\left[x_i(k_p) - x_j(k_p)\right]$ for $i \in \mathcal{U}_c$; because we assume a fully connected graph, $\alpha_{r^*}^{j \to r^*} \leqslant \gamma\left[x_j(k_p) - x_{r^*}(k_p)\right]$ for $j \in \mathcal{U}_c$. Thus, by applying these two equations for Equation (45) and using the fact that $\sum_{r',r' \neq i} \alpha_{r'}^{r' \to j} \leqslant 0$, we obtain

$$\begin{aligned}
x_j(k_p + 1) &\leqslant x_{r^*}(k_p) + \alpha_{r^*}^{j \to r^*} + \sum_{r',r' \neq i} \alpha_{r'}^{r' \to j} + \alpha_j^{i \to j} \\
&\leqslant x_{r^*}(k_p) + \gamma\left[x_i(k_p) - x_j(k_p)\right] + \gamma\left[x_j(k_p) - x_{r^*}(k_p)\right] \\
&= x_{r^*}(k_p) + \gamma\left[x_i(k_p) - x_{r^*}(k_p)\right] \\
&\leqslant \max_{i \in \mathcal{I}}\{x_i(k)\} + \gamma\left[x_i(k_p) - \max_{i \in \mathcal{I}}\{x_i(k)\}\right]
\end{aligned} \tag{47}$$

By applying Equation (38) to $x_i(k_p)$ in Equation (47), we have

$$\begin{aligned}
x_j(k_p + 1) &\leqslant \max_{i \in \mathcal{I}}\{x_i(k)\} + \gamma\Bigg[\max_{i \in \mathcal{I}}\{x_i(k)\} \\
&\quad + \gamma^{k_p - k}\left[x_i(k) - \max_{i \in \mathcal{I}}\{x_i(k)\}\right] - \max_{i \in \mathcal{I}}\{x_i(k)\}\Bigg] \\
&= \max_{i \in \mathcal{I}}\{x_i(k)\} + \gamma^{k_p - k + 1}\left[x_i(k) - \max_{i \in \mathcal{I}}\{x_i(k)\}\right]
\end{aligned} \tag{48}$$

If we apply Equation (38) to x_j with $k = k_p + 1$ and $t = k' - k_p - 1$, we obtain

$$\begin{aligned}
x_j(k') &\leqslant \max_{i \in \mathcal{I}}\{x_i(k_p + 1)\} + \gamma^{k' - k_p - 1}\left[x_j(k_p + 1) - \max_{i \in \mathcal{I}}\{x_i(k_p + 1)\}\right] \\
&\leqslant \max_{i \in \mathcal{I}}\{x_i(k)\} + \gamma^{k' - k_p - 1}\Bigg[\max_{i \in \mathcal{I}}\{x_i(k)\} + \gamma^{k_p - k + 1}[x_i(k) \\
&\quad - \max_{i \in \mathcal{I}}\{x_i(k)\}\Bigg] - \max_{i \in \mathcal{I}}\{x_i(k)\}\Bigg] \\
&= \max_{i \in \mathcal{I}}\{x_i(k)\} + \gamma^{k' - k}\left[x_i(k) - \max_{i \in \mathcal{I}}\{x_i(k)\}\right], \ \forall\, k' \geqslant k_p + 1
\end{aligned} \tag{49}$$

– Let us consider time k_p, $p \in \{1, 2 \ldots\}$, and $j' \in \mathcal{N}_i$ such that $x_{j'}(k_p) \leqslant x_i(k_p)$, that is, inverter i does not pass a non-negligible amount of capacitive reactive power to inverter j'. In this case, it is obvious from Equation (38) with $k = k_p$ and $t = k' - k_p$ that

$$x_{j'}(k') \leqslant \max_{i \in \mathcal{I}}\{x_i(k_p)\} + \gamma^{k'-k_p}\left[x_j(k_p) - \max_{i \in \mathcal{I}}\{x_i(k_p)\}\right]$$

$$\leqslant \max_{i \in \mathcal{I}}\{x_i(k)\} + \gamma^{k'-k_p}\left[x_i(k_p) - \max_{i \in \mathcal{I}}\{x_i(k)\}\right] \quad (50)$$

for all $k' \geqslant k_p$. From Equation (38) with $t = k_p - k$, it is also clear that

$$x_i(k_p) \leqslant \max_{i \in \mathcal{I}}\{x_i(k)\} + \gamma^{k_p-k}\left[x_i(k) - \max_{i \in \mathcal{I}}\{x_i(k_p)\}\right] \quad (51)$$

It follows from Equations (50) and (51) that

$$x_{j'}(k') \leqslant \max_{i \in \mathcal{I}}\{x_i(k)\} + \gamma^{k'-k_p}\left[\max_{i \in \mathcal{I}}\{x_i(k)\} + \gamma^{k_p-k}\left[x_i(k)\right.\right.$$

$$\left.\left. - \max_{i \in \mathcal{I}}\{x_i(k_p)\}\right] - \max_{i \in \mathcal{I}}\{x_i(k)\}\right] \quad (52)$$

$$= \max_{i \in \mathcal{I}}\{x_i(k)\} + \gamma^{k'-k}\left[x_i(k) - \max_{i \in \mathcal{I}}\{x_i(k)\}\right], \ \forall \ k' \geqslant k_p$$

Notice that at each time k_p, $p \in \{1, 2, \ldots\}$, for any $i \in \mathcal{U}_c$ and any $j \in \mathcal{U}_c$, one of the two cases shown previously must be valid for a fully connected graph. Also, for certain $i \in \mathcal{U}_c$ and certain $j \in \mathcal{U}_c$, one of the two cases must occur every B steps. Hence, if we choose $k_p = k_1$ and $k' > k_p$, Equations (50) and (52) indicate that Equation (40) is valid for all $k' \geqslant k+B$, $j \in \mathcal{N}_i$. Also, because we assume a fully connected graph and $\max_{i \in \mathcal{U}_c}\{x_i(k')\} = \max_{i \in \mathcal{I}}\{x_i(k')\}$, Equation (40) is turned into Equation (41). As we made no assumptions to the contrary, Equation (41) is valid for any $i \in \mathcal{U}_c$. Hence, we can replace $x_i(k)$ with $\min_{i \in \mathcal{U}_c}\{x_i(k)\}$, and Equation (41) becomes

$$\max_{i \in \mathcal{I}}\{x_i(k')\} \leqslant \max_{i \in \mathcal{I}}\{x_i(k)\} + \gamma^{k'-k}\left[\min_{i \in \mathcal{U}_c}\{x_i(k)\} - \max_{i \in \mathcal{I}}\{x_i(k)\}\right], \ \forall \ k' \geqslant k + B \quad (53)$$

- Next, fix $i \in \mathcal{S}_c$ and $k \geqslant 0$, similar to the analysis for Equation (38), we obtain from condition (2a) that

$$x_i(k + t) \leqslant q_i^{\max} + \gamma^t\left[x_i(k) - q_i^{\max}\right], \ \forall \ i \in \mathcal{S}_c \quad (54)$$

  - Similar to the analysis for inverter $i \in \mathcal{U}_c$, we consider a time k_p, $p \in \{1, 2, \ldots\}$, and $j \in \mathcal{N}_i$ such that $x_j(k_p) > x_i(k_p)$, that is, at time k_p, inverter i passes a non-negligible amount of capacitive reactive power to inverter j. Equations (42)–(46) also apply to $j \in \mathcal{N}_i$. Because $i \in \mathcal{S}_c$, according to condition (a3), we know that $\alpha_j^{i \to j} \leqslant \gamma \max\left\{\left[x_i(k_p) - x_j(k_p)\right], \left[x_i(k_p) - q_i^{\max}\right]\right\}$

    * Let us consider the case that $\left[x_i(k_p) - x_j(k_p)\right] \geqslant \left[x_i(k_p) - q_i^{\max}\right]$, that is, $x_j(k_p) \leqslant q_i^{\max}$. Then, $\alpha_j^{i \to j} \leqslant \gamma\left[x_i(k_p) - x_j(k_p)\right]$. Consider $j \in \mathcal{U}_c$, then $\alpha_{r*}^{j \to r*} \leqslant \gamma\left[x_j(k_p) - x_{r*}(k_p)\right]$, the following analysis is the same as the one for the case that $i \in \mathcal{U}_c$, and we directly obtain the same result as Equation (49). As $\max_{i \in \mathcal{I}}\{x_i(k)\} > q_i^{\max}$ for any $i \in \mathcal{S}_c$, Equation (49) is turned into

$$x_j(k') \leqslant \max_{i \in \mathcal{I}}\{x_i(k)\} + \gamma^{k'-k}\left[x_i(k) - q_i^{\max}\right], \ \forall \ k' \geqslant k_p + 1 \quad (55)$$

    * Let us consider the case that $\left[x_i(k_p) - x_j(k_p)\right] < \left[x_i(k_p) - q_i^{\max}\right]$, that is, $x_j(k_p) > q_i^{\max}$. Then, $\alpha_j^{i \to j} \leqslant \gamma\left[x_i(k_p) - q_i^{\max}\right]$. Consider $j \in \mathcal{U}_c$, using the fact that $\alpha_{r*}^{j \to r*} \leqslant 0$ and $\sum_{r', r' \neq i} \alpha_{r'}^{r' \to j} \leqslant 0$ Equation (47) is rewritten as

$$
\begin{aligned}
x_j(k_p + 1) &\leqslant x_{r^*}(k_p) + \alpha_j^{i \to j} \\
&\leqslant \max_{i \in \mathcal{I}}\{x_i(k)\} + \gamma\left[x_i(k_p) - q_i^{\max}\right]
\end{aligned}
\tag{56}
$$

By applying Equation (54) to $x_i(k_p)$ in Equation (56) with $t = k_p - k$, we arrive at

$$
\begin{aligned}
x_j(k_p + 1) &\leqslant \max_{i \in \mathcal{I}}\{x_i(k)\} + \gamma\left[q_i^{\max} + \gamma^{k_p - k}\left[x_i(k) - q_i^{\max}\right] - q_i^{\max}\right] \\
&\leqslant \max_{i \in \mathcal{I}}\{x_i(k)\} + \gamma^{k_p - k + 1}\left[x_i(k) - q_i^{\max}\right]
\end{aligned}
\tag{57}
$$

If we apply Equation (38) to x_j with $k = k_p + 1$ and $t = k' - k_p - 1$, we obtain

$$
\begin{aligned}
x_j(k') &\leqslant \max_{i \in \mathcal{I}}\{x_i(k_p + 1)\} + \gamma^{k' - k_p - 1}\left[x_j(k_p + 1) - \max_{i \in \mathcal{I}}\{x_i(k_p + 1)\}\right] \\
&\leqslant \max_{i \in \mathcal{I}}\{x_i(k)\} + \gamma^{k' - k_p - 1}\left[q_i^{\max} + \gamma^{k_p - k + 1}\left[x_i(k) - q_i^{\max}\right] - q_i^{\max}\right] \\
&= \max_{i \in \mathcal{I}}\{x_i(k)\} + \gamma^{k' - k}\left[x_i(k) - q_i^{\max}\right], \ \forall \, k' \geqslant k_p + 1
\end{aligned}
\tag{58}
$$

– Let us consider time k_p, $p \in \{1, 2, \ldots, \}$ and $j' \in \mathcal{N}_i$ such that inverter $i \in \mathcal{S}_c$ does not pass a non-negligible amount of capacitive reactive power to inverter j'. It is either the case that $x_{j'}k_p < x_i(k_p)$ or $x_i(k_p) = q_i^{\max}$. If it is the case that $x_{j'}k_p < x_i(k_p)$, by conducting a similar analysis to the case that $i \in \mathcal{U}_c$ and $x_{j'}k_p < x_i(k_p)$, we can derive the same result as Equation (58). If it is the case that $x_i(k_p) = q_i^{\max}$, it is obvious that Equation (58) still holds.

As Equation (58) is valid for all $j \in \mathcal{U}_c$, we have

$$
\max_{i \in \mathcal{U}_c}\{x_i(k)\} \leqslant \max_{i \in \mathcal{I}}\{x_i(k)\} + \gamma^{k' - k}\left[x_i(k) - q_i^{\max}\right], \ \forall \, k' \geqslant k_p + 1
\tag{59}
$$

Because every B time steps at least one of the two cases discussed previously must occur, Equation (59) must be valid for all $k' \geqslant k + B$. Also, because we made no contrary to x_i for any $i \in \mathcal{S}_c$, Equation (58) is modified to Equation (60):

$$
\max_{i \in \mathcal{U}_c}\{x_i(k)\} \leqslant \max_{i \in \mathcal{I}}\{x_i(k)\} + \gamma^{k' - k} \min_{i \in \mathcal{S}_c}\{x_i(k) - q_i^{\max}\}, \ \forall \, k' \geqslant k + B
\tag{60}
$$

By adding Equation (60) to Equation (53), we obtain

$$
\begin{aligned}
\max_{i \in \mathcal{I}}\{x_i(k')\} \leqslant{}& \max_{i \in \mathcal{I}}\{x_i(k)\} + \frac{\gamma^{k' - k}}{2}\left[\min_{i \in \mathcal{U}_c}\{x_i(k)\}\right. \\
&\left. - \max_{i \in \mathcal{I}}\{x_i(k)\} + \min_{i \in \mathcal{S}_c}\{x_i(k) - q_i^{\max}\}\right], \ \forall \, k' \geqslant k + B
\end{aligned}
\tag{61}
$$

Using the fact that $x_i(k) \leqslant q_i^{\max}$ for all $i \in \mathcal{S}_c$, Equation (61) implies that

$$
\begin{aligned}
&\max_{i \in \mathcal{I}}\{x_i(k)\} - \max_{i \in \mathcal{I}}\{x_i(k')\} \\
\geqslant{}& \frac{\gamma^{k' - k}}{2}\left[\max_{i \in \mathcal{I}}\{x_i(k)\} - \min_{i \in \mathcal{U}_c}\{x_i(k)\} - \min_{i \in \mathcal{S}_c}\{x_i(k) - q_i^{\max}\}\right] \\
={}& \frac{\gamma^{k' - k}}{2}\left[\max_{i \in \mathcal{I}}\{x_i(k)\} - \min_{i \in \mathcal{U}_c}\{x_i(k)\} + \max_{i \in \mathcal{S}_c}|x_i(k) - q_i^{\max}|\right], \ \forall \, k' \geqslant k + B
\end{aligned}
\tag{62}
$$

Notice that at any moment $\max_{i\in\mathcal{I}}\{x_i(k)\} = \max_{i\in\mathcal{U}_c}\{x_i(k)\}$. By applying Equation (32) to $V(x(k))$ and $V(x(k'))$, we have

$$
\begin{aligned}
&V(x(k)) - V(x(k')) \\
&= \max_{i\in\mathcal{I}}\{x_i(k)\} - \max_{i\in\mathcal{I}}\{x_i(k')\} - \frac{1}{|\mathcal{U}_c|}\sum_{j\in\mathcal{U}_c} x_j(k) \\
&\quad + \frac{1}{|\mathcal{U}_c|}\sum_{i\in\mathcal{S}_c}|x_i(k) - q_i^{\max}| + \frac{1}{|\mathcal{U}_c|}\sum_{j\in\mathcal{U}_c} x_j(k') - \frac{1}{|\mathcal{U}_c|}\sum_{i\in\mathcal{S}_c}|x_i(k') - q_i^{\max}| \\
&= \max_{i\in\mathcal{I}}\{x_i(k)\} - \max_{i\in\mathcal{I}}\{x_i(k')\} - \frac{1}{|\mathcal{U}_c|}\sum_{j\in\mathcal{U}_c} x_j(k) - \frac{1}{|\mathcal{U}_c|}\sum_{i\in\mathcal{S}_c} x_i(k) \\
&\quad + \frac{1}{|\mathcal{U}_c|}\sum_{i\in\mathcal{S}_c} q_i^{\max} + \frac{1}{|\mathcal{U}_c|}\sum_{j\in\mathcal{U}_c} x_j(k') + \frac{1}{|\mathcal{U}_c|}\sum_{i\in\mathcal{S}_c} x_i(k') - \frac{1}{|\mathcal{U}_c|}\sum_{i\in\mathcal{S}_c} q_i^{\max}
\end{aligned}
\tag{63}
$$

It is obvious that $\frac{1}{|\mathcal{U}_c|}\sum_{j\in\mathcal{U}_c} x_j(k) + \frac{1}{|\mathcal{U}_c|}\sum_{i\in\mathcal{S}_c} x_i(k)$ is consistent with time. Hence, by applying Equation (62) and Equation (33), Equation (63) is turned into

$$
\begin{aligned}
V(x(k)) - V(x(k')) &= \max_{i\in\mathcal{I}}\{x_i(k)\} - \max_{i\in\mathcal{I}}\{x_i(k')\} \\
&\geqslant \frac{\gamma^{k'-k}}{2}\left[\max_{i\in\mathcal{I}}\{x_i(k)\} - \min_{i\in\mathcal{U}_c}\{x_i(k)\} + \max_{i\in\mathcal{S}_c}|x_i(k) - q_i^{\max}|\right] \\
&\geqslant \frac{\gamma^{k'-k}}{2}\rho(x(k), \mathcal{X}_c^+), \ \forall\, k' \geqslant k + B
\end{aligned}
\tag{64}
$$

Equation (36) indicates that $c_1\rho(x(k), \mathcal{X}_c^+) \leqslant V(x(k)) \leqslant c_2\rho(x(k), \mathcal{X}_c^+)$, where $c_1 = \frac{1}{|\mathcal{U}_c|}$ and $c_2 = (2 + |\mathcal{S}_c|)$, and Equation (63) indicates that $V(x(k)) - V(x(k')) \geqslant c_3\rho(x(k), \mathcal{X}_c^+)$ for all $k' \geqslant k + B$, where $c_3 = \frac{\gamma^{k'-k}}{2}$. It is obvious that $\frac{c_3}{c_2} \in (0, 1)$. Hence, the invariant set $\mathcal{X}_c^+$ is exponentially stable in large. Similarly, we can prove that the invariant set $\mathcal{X}_c^-$ is exponentially stable in large for the case when $Q_D < 0$. Hence, with Assumptions 1 and 2, and the reactive power passing conditions (a1)–(a3) and (b1)–(b3), the invariant set $\mathcal{X}_c$ is exponentially stable in large.

Proof of Lemma 3
The proof of Lemma 3 is similar to Lemma 1 and is not presented here.

Proof of Lemma 4
The proof of Lemma 4 is similar to Lemma 2 and is not presented here.

Proof of Theorem 3
The proof of Theorem 3 is similar to Theorem 1 and is not presented here.

Proof of Theorem 4
The proof of Theorem 4 is similar to Theorem 2 and is not presented here.

Proof of Theorem 5
The proof of Theorem 5 is similar to Theorem 1 and is not presented here.

Proof of Theorem 6
The proof of Theorem 6 is similar to Theorem 2 and is not presented here.

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