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Stable adaptive control of feedback linearizable time-varying non-linear systems with application to fault-tolerant engine control

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Stable indirect and direct adaptive controllers are presented for a class of input–output feedback linearizable time-varying non-linear systems. The radial basis function neural networks are used as on-line approximators to learn the time-varying characteristics of system parameters. Stability results are given in the paper, and the performance of the indirect and direct adaptive schemes is demonstrated through a fault-tolerant engine control problem where the faults are naturally time-varying.

1. Introduction

Adaptive control has been employed in situations where little a priori knowledge of the plant is known. Adaptive control has also been used to compensate for on-line system parameter variations, which may arise due to changes in operating points, component faults, plant deterioration, etc. The general methodology of adaptive control for time-varying systems is to treat the effects of parameter variations as un-modelled perturbations so that it turns into a robustness problem (Ioannou and Sun 1996). This methodology has been applied in linear time-varying systems, where the parameters vary slowly and smoothly, or discontinuously (i.e. jumps) but the discontinuities occur over large intervals of time (Middleton and Goodwin 1988, Wen 1994, Watkins and Kiriakidis 1998, Zhang and Chai 1998). In a monograph Tsakalis and Ioannou (1993) presented a major work on the topic of adaptive control for linear time-varying systems using model reference adaptive control or adaptive pole placement control schemes, which also appeared in earlier publications (Tsakalis and Ioannou 1987, 1989, 1990, 1992). The assumption of slow parameter variations may also be relaxed if some information about the rapidly varying parameters is available a priori (Marino and Tomei 1998, 1999).

Adaptive control for non-linear time varying systems has also been studied by some researchers, but only restricted classes of systems are considered and only limited results exist so far. In Tao (1991) a high-order non-linear plant was remodelled to be a lower-order linear time-invariant plant with an uncertainty which is bounded by a bounding signal, and global stability and minimal tracking error were guaranteed using reduced-order adaptive feedback controllers with regular or variable structure. Marino and Tomei (1993) designed a robust state feedback control for a class of non-linear time-varying systems

$$\begin{aligned} \dot{x} &= \bar{f}(x, \theta(t)) + g(x)u \\ &= f(x) + q(x, \theta(t)) + g(x)u \\ y &= h(x) \end{aligned}$$

with unknown unmodelled time-varying parameters (or disturbances) with $\theta(t)$ whose bounds are known. It was assumed that the nominal system (f, g) is locally (globally) feedback-linearizable and that the uncertain term $q(x, \theta(t))$ satisfies triangularity conditions so that the above non-linear system can be transformed into the strict feedback form

$$\begin{aligned} \dot{z}_1 &= z_2 + \phi_1(z_1, \theta(t)) \\ \dot{z}_2 &= z_3 + \phi_2(z_1, z_2, \theta(t)) \\ &\vdots \\ \dot{z}_n &= v + \phi_n(z_1, \dots, z_n, \theta(t)). \end{aligned}$$

This result was extended to adaptive control in Marino and Tomei (1998) (and Marino and Tomei (1997) by adaptive output feedback control) but the class of non-linear systems is restricted to be linear with respect to unknown time-varying parameters

$$\begin{aligned} \dot{x} &= f(x) + g(x)u + \sum_{i=1}^{p} \theta_i(t)\, q_i(x) + \sum_{j=1}^{s} d_j(t)\, r_j(x) \\ y &= h(x). \end{aligned}$$

The class of non-linear time-varying systems in the strict feedback form was also studied in Wu and Chou (1999) and Lin (1997) using the backstepping design method.

In addition, another class of time-varying non-linear systems with an underlying strict feedback structure has also been considered, and the control law is designed using the backstepping methodology (Ordonez and

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Passino 1999, 2000 a). This class of systems consists of an interpolation of non-linear dynamic equations

$$\dot{x}_i = \sum_{j=1}^{R} \rho_j(v)\big(\phi_i^j(X_i) + \psi_i^j(X_i)x_{i+1}\big)$$
$$\dot{x}_n = \sum_{j=1}^{R} \rho_j(v)\big(\phi_n^j(X_n) + \psi_n^j(X_n)u\big)$$

where $X_i = [x_1, \ldots, x_i]^\top$ and v is an exogenous scheduling variable. The result has also been extended to both indirect and direct adaptive control (Ordonez and Passino 2000b, c).

In this paper we consider a more general class of non-linear time-varying systems, which is input–output feedback-linearizable, and present stable adaptive control approaches using the on-line learning capabilities of radial basis function neural networks. This class of systems is large enough so that it is not only of theoretical interest but also of practical applicability. Meanwhile, on-line approximation-based stable adaptive neural/fuzzy methods, as covered in a recent text (Spooner *et al.* 2001), have been widely used in adaptive control for non-linear systems and been significantly impacted by the work in Narendra and Parthasarathy (1990), Polycarpou (1991), Polycarpou and Ioannou (1991), Sanner and Slotine (1992), Yabuta and Yamada (1992), Liu and Chen (1993), Sadegh (1993), Chen and Liu (1994), Rovithakis and Christodoulou (1994), Yeşildirek and Lewis (1995), Fabri and Kadirkamanathan (1996), Farrel (1996), Lewis *et al.* (1996) and Polycarpou and Mears (1998) using neural networks as approximators of non-linear functions; the work in Su and Stepanenko (1994), Wang (1994a, b), Hsu and Fu (1995), Chen *et al.* (1996), Lee and Wang (1996) and Spooner and Passino (1996) using fuzzy systems for the same purpose; and the work in Narendra and Parthasarathy (1990) and Rovithakis and Christodoulou (1994) using dynamical neural networks. The neural and fuzzy approaches are most of the time equivalent, differing between each other mainly in the structure of the approximator chosen. Indeed, to try to bridge the gap between the neural and fuzzy approaches several researchers (e.g., in Spooner and Passino 1996) introduce adaptive schemes using a class of parameterized functions that include both neural networks and fuzzy systems. As to the approximator structure, linear parameter approximators are used in Polycarpou and Ioannou (1991), Sanner and Slotine (1992), Sadegh (1993), Su and Stepanenko (1994), Carelli *et al.* (1995), Hsu and Fu (1995), Chen *et al.* (1996), Farrel (1996), Fabri and kadirkananathan (1996), Polycarpou (1996) and Spooner and Passino (1996) and non-linear in Narendra Parthasarathy (1990), Yabuta and Yamada (1992), Liu and Chen (1993), Chen and Liu (1994), Yeşildirek and Lewis (1995), Lewis *et al.* (1996) and Polycarpou and Mears (1998). Finally, most of the papers deal with indirect adaptive control, whereas very few authors face the direct approach (see, however, Rovithakis and Chistodoulau 1995, Spooner and Passino 1996).

This paper is organized as follows. The spatially localized model architecture of radial basis function networks is discussed in § 2, and in § 3 the details of the problem formulation for input–output feedback linearizable time-varying non-linear systems are given. The adaptive algorithms and system stability analysis are presented in § 4 and 5 for the indirect and direct cases. At the end of those sections we will comment on the relationships between the work here and the most relevant work discussed above. In this way we will further clarify the theoretical contribution of this paper. Simulation examples for a fault-tolerant engine control problem are given in § 6 to demonstrate the effectiveness of the proposed adaptive schemes. Actually, the jet engine provided the motivation for this research: faults are naturally time-varying phenomena so that existing on-line approximation-based approaches were limited in their applicability and hence the application dictated the need to generalize existing approaches to the time-varying case.

2. Radial basis function neural networks

In neurobiological studies, the concept of localized information processing in the form of receptive fields has been known and demonstrated by experimental evidence (e.g. locally tuned and overlapping receptive fields have been found in parts of the cerebral cortex, in the visual cortex, and in other parts of the brain), which suggests that such local learning offers alternative computational opportunities to learning with 'global basis functions', such as the multilayer perceptron neural network with sigmoidal activation functions (Schaal and Atkeson 1998). Inspired by these biological counterparts, the radial basis function neural network model has been presented, which can be defined by

$$y = F_{rbf}(x, \theta) = \sum_{i=1}^{M} b_i R_i(x) \tag{1}$$

where y is the output of the radial basis function network, $x = [x_1, x_2, \ldots, x_n]^\top$ holds the n inputs, $i = 1, 2, \ldots, M$ represent M receptive field units, and θ holds the parameters of the 'receptive field units', which consist of the 'strength' parameters b_i and possibly the parameters of the 'radial basis functions' $R_i(x)$. There are several possible choices for the receptive field functions $R_i(x)$. Typically, Gaussian-shaped functions are used for analytical convenience, that is

$$R_i(x) = \exp\left(-\frac{1}{2}(x - c_i)^\top D_i(x - c_i)\right) \tag{2}$$

where $c_i = [c_1^i, c_2^i, \ldots, c_n^i]^\top$ parameterize the locations of the receptive fields in the input space, and

$$D_i = \mathrm{diag}\left(\left(\frac{1}{\sigma_1^i}\right)^2, \left(\frac{1}{\sigma_2^i}\right)^2, \ldots, \left(\frac{1}{\sigma_n^i}\right)^2\right)$$

determine the shapes (or relative widths) of the receptive fields. Note that rather than computing the output of the radial basis function network with the simple sum as in (1), there are also alternatives, for instance, by computing a weighted average

$$y = F_{rbf}(x, \theta) = \frac{\sum_{i=1}^M b_i R_i(x)}{\sum_{i=1}^M R_i(x)}. \tag{3}$$

Moreover, it is also possible to further define the strength parameters b_i to be parametric functions

$$b_i(x) = a_{i,0} + a_{i,1}\, x_1 + \cdots + a_{i,n}\, x_n \tag{4}$$

where $a_{i,j}$, $i = 1, 2, \ldots, M$ and $j = 1, 2, \ldots, n$, are strength function parameters, so as to improve the modeling flexibility of the radial basis function network.

By having the tunable parameter vector θ composed of strength function parameters $a_{i,j}$ only, and specifying the radial basis function parameters c_i and D_i in advance, we will have a *linear in the parameter* radial basis function network

$$y = F_{rbf}(x, \theta) = \theta^\top \phi(x). \tag{5}$$

Note that the *linear in the parameter* radial basis function networks also have capabilities of forming an arbitrarily accurate approximation to any continuous non-linear function, so that in the following adaptive mechanisms we use them as on-line approximators to learn the time-varying dynamics of the system. This will facilitate the derivation of adaptive laws and the analysis of system stability. It is worth mentioning that even though radial basis networks are used in this paper as on-line approximators, other *linear in the parameter* approximators such as B-spline neural networks and Takagi–Sugeno fuzzy systems, are also applicable.

3. Input–output feedback linearizable time-varying non-linear systems

Consider the following single-input single-output non-linear time-varying system

$$\dot{x} = f(x, t) + g(x, t)u \tag{6}$$

$$y = h(x, t) \tag{7}$$

where $x = [x_1, x_2, \ldots, x_n]^\top$ is the state vector, u is the (scalar) input, and y is the (scalar) output of the system. The functions $f\colon D \times [0, \infty) \to \Re^n$, $g\colon D \times [0, \infty) \to \Re^n$ and $h\colon D \times [0, \infty) \to \Re$ are smooth time-varying functions, and $D \subset \Re^n$ is a domain that contains the origin $x = 0$. For convenience we assume that if $u = 0, \forall t \geq 0$, the origin is an equilibrium point at $t = 0$ and for all subsequent times, that is, $f(0, t) = 0, \forall t \geq 0$. There is no loss of generality in doing so because any non-zero equilibrium point (or, more generally, a non-zero solution of the system) can be transformed to the origin via a change of variables. To see this, suppose $\bar{x}(t)$ is a solution of the system $\dot{x} = f(x, t)$ defined for all $t \geq 0$, and consider the change of variables $z = x - \bar{x}(t)$. We have

$$\begin{aligned}\dot{z} &= f(x, t) - \dot{\bar{x}}(t)\\ &= f(z + \bar{x}(t), t) - f(\bar{x}(t), t)\\ &= \bar{f}(z, t)\end{aligned}$$

where if $z = 0, \bar{f}(0, t) = 0$ for all $t \geq 0$; that is, in the new variable z, the system has equilibrium at the origin. Therefore, we may determine the stability of the solution of the original system by studying the stability of the origin as an equilibrium point for the transformed system.

Note that the standard Lie derivative and strong relative degree for time-invariant systems (Khalil 1996) are not adequate for time-varying systems, and modifications need to be made to explicitly account for the time variability of the system. Let $\bar{L}_f^d h(x, t)$ be the dth Lie derivative of $h(x, t)$ with respect to $f(x, t)$. In particular, define

$$\bar{L}_f\, h(x, t) = \frac{\partial h}{\partial t} + \left(\frac{\partial h}{\partial x}\right)^\top f(x, t) \tag{8}$$

and, for example

$$\bar{L}_f^2 h(x, t) = \bar{L}_f\,[\bar{L}_f\, h(x, t)] = \frac{\partial \bar{L}_f\, h}{\partial t} + \left(\frac{\partial \bar{L}_f\, h}{\partial x}\right)^\top f(x, t). \tag{9}$$

Note that the *modified* Lie derivative $\bar{L}_f^d h(x, t)$ for time-varying systems is a straightforward extension of the standard Lie derivative $L_f^d h(x)$ (so that no specific definition is typically given in differential geometry). Particularly, define $x_0 = t$ and consider the 'extended' vectors $X = [x_0, x_1, x_2, \ldots, x_n]^\top$ and $F(X) = [1, f(X)^\top]^\top$ (note that $\dot{x}_0 = \dot{t} = 1$), so that we have

$$L_F h(X) = \left(\frac{\partial h}{\partial X}\right)^\top F(X) = \frac{\partial h}{\partial t} + \left(\frac{\partial h}{\partial x}\right)^\top f(x, t) = \bar{L}_f\, h(x, t).$$

Next, we define the 'strong relative degree' of the time-varying system. A system is said to have a strong relative degree d, $1 \leq d \leq n$, in a region $D_0 \subset D$ if

$$L_g h(x, t) = L_g \bar{L}_f\, h(x, t) = \cdots = L_g \bar{L}_f^{d-2} h(x, t) = 0 \tag{10}$$

and

$$L_g \bar{L}_f^{d-1} h(x,t) \neq 0 \tag{11}$$

for all $x \in D_0$ and $t \in [0, \infty)$. Note that we use both the standard and the modified Lie derivatives above to provide a compact representation of the definition.

Under the above definitions, if the system defined by (6) and (7) has a strong relative degree d, then

$$\begin{aligned}
\dot{y} &= \frac{\partial h}{\partial t} + \left(\frac{\partial h}{\partial x}\right)^{\top} [f(x,t) + g(x,t)u] \\
&= \left[\frac{\partial h}{\partial t} + \left(\frac{\partial h}{\partial x}\right)^{\top} f(x,t)\right] + \left[\left(\frac{\partial h}{\partial x}\right)^{\top} g(x,t)\right] u \\
&= \bar{L}_f h(x,t) \\
\ddot{y} &= \frac{\partial \bar{L}_f h}{\partial t} + \left(\frac{\partial \bar{L}_f h}{\partial x}\right)^{\top} [f(x,t) + g(x,t)u] \\
&= \left[\frac{\partial \bar{L}_f h}{\partial t} + \left(\frac{\partial \bar{L}_f h}{\partial x}\right)^{\top} f(x,t)\right] + \left[\left(\frac{\partial \bar{L}_f h}{\partial x}\right)^{\top} g(x,t)\right] u \\
&= \bar{L}_f^2 h(x,t)
\end{aligned}$$

and so on, so that the system dynamics may be written in the normal form as

$$\begin{aligned}
\dot{\xi}_1 &= \xi_2 = \bar{L}_f h(x,t) \\
\dot{\xi}_2 &= \xi_3 = \bar{L}_f^2 h(x,t) \\
&\vdots \\
\dot{\xi}_{d-1} &= \xi_d = \bar{L}_f^{d-1} h(x,t) \\
\dot{\xi}_d &= \bar{L}_f^d h(x,t) + L_g \bar{L}_f^{d-1} h(x,t) u = \alpha(\xi,\pi,t) + \beta(\xi,\pi,t) u \\
\dot{\pi} &= f_0(\xi,\pi,t)
\end{aligned}$$

with $\xi \in \Re^d$, $\pi \in \Re^{n-d}$, and $\xi_1 = y$. This transformation of the model form can be taken by a change of variables

$$z = T(x,t) = \begin{bmatrix} \psi_1(x,t) \\ \vdots \\ \psi_d(x,t) \\ \cdots\cdots \\ \phi_1(x,t) \\ \vdots \\ \phi_{n-d}(x,t) \end{bmatrix} = \begin{bmatrix} \psi(x,t) \\ \cdots\cdots \\ \phi(x,t) \end{bmatrix} = \begin{bmatrix} \xi \\ \cdots \\ \pi \end{bmatrix} \tag{12}$$

where $\psi(x,t)$ and $\phi(x,t)$ are given by

$$\psi_1(x,t) = h(x,t) \tag{13}$$

$$\psi_{i+1}(x,t) = \frac{\partial \psi_i}{\partial t} + \left(\frac{\partial \psi_i}{\partial x}\right)^{\top} f(x,t) \tag{14}$$

$$\left(\frac{\partial \phi_j}{\partial x}\right)^{\top} g(x,t) = 0 \tag{15}$$

for $i = 1, 2, \ldots, d-1$, $j = 1, 2, \ldots, n-d$, $\forall x \in D_x$ and $\forall t \in [0, \infty)$, so that $T(x,t)$ is a diffeomorphism (both the map $T(\cdot)$ and its inverse $T^{-1}(\cdot)$ are continuously differentiable) on a domain $D_x \subset D_0$. It has been shown in Palanki and Kravaris (1997) that if the system (6) and (7) has relative degree $d < n$ in a domain D_0 for all $t \in [0, \infty)$, then there exist functions ϕ_1 to ϕ_{n-d} that satisfy (15) and make $T(x,t)$ a diffeomorphism. A 'global' normal form can be obtained if and only if the conditions (13), (14) and (15) hold for all $x \in \Re^n$ and T is proper, that is, $\lim_{\|x\| \to \infty} \|T(x,t)\| = \infty$.

For convenience, we assume that $x = 0$ is an equilibrium point of the original system and y vanishes at $x = 0$, that is, $f(0,t) = 0$ and $h(0,t) = 0$, $\forall t \geq 0$. By choosing $\phi(x,t)$ such that $\phi(0,t) = 0, \forall t \geq 0$, the equilibrium point of the transformed system is defined by $\bar{\xi} = [h(0,t), 0, \ldots, 0]^{\top} = 0$ and $\bar{\pi} = \phi(0,t) = 0$.

The normal form decomposes the system states into an external part ξ and an internal part π. For the external part, if we let $y^{(d)}$ denote the dth derivative of y, it may be rewritten as

$$y^{(d)} = [\alpha_k(t) + \alpha(x,t)] + [\beta_k(t) + \beta(x,t)]u \tag{16}$$

where $\alpha_k(t)$ and $\beta_k(t)$ are 'known' dynamics of the system (which are assumed to be bounded if x is bounded), and $\alpha(x,t)$ and $\beta(x,t)$ represent non-linear time-varying dynamics of the plant that are unknown. We assume that for some known $\beta_0 > 0$, we have $|\beta_k(t) + \beta(x,t)| \geq \beta_0$ so that it is always bounded away from zero (for convenience we further assume that $\beta_k(t) + \beta(x,t) > 0$, however, the following analysis may easily be modified for systems which are defined with $\beta_k(t) + \beta(x,t) < 0$). The external part may be stabilized by the control u (which we will show later), while the internal part is made uncontrollable by the same control. By having $\xi = 0$ in the inner part, the 'zero dynamics' of the system are given by

$$\dot{\pi} = f_0(0,\pi,t). \tag{17}$$

If the plant is of relative degree $d = n$, then there are no zero dynamics. Alternatively, if the relative degree $d < n$, we assume that the zero dynamics are uniformly exponentially attractive so that we have

$$\gamma_1 \|\pi\|^2 \leq v_1(\pi,t) \leq \gamma_2 \|\pi\|^2 \tag{18}$$

$$\frac{\partial v_1}{\partial t} + \left(\frac{\partial v_1}{\partial \pi}\right)^{\top} f_0(0,\pi,t) \leq -\gamma_3 \|\pi\|^2 \tag{19}$$

$$\left\|\frac{\partial v_1}{\partial \pi}\right\| \leq \gamma_4 \|\pi\| \tag{20}$$

for some positive constants γ_1, γ_2, γ_3, and γ_4. If $f_0(0,\pi,t)$ is Lipschitz in ξ, then $\|f_0(\xi,\pi,t) - f_0(0,\pi,t)\| \leq k_1 \|\xi\|$ for some positive k_1. Now, if we have some control u so that $\|\xi\| \leq k_2$ where k_2 is some positive constant, we have

$$\dot{v}_1 = \frac{\partial v_1}{\partial t} + \left(\frac{\partial v_1}{\partial \pi}\right)^\top f_0(\xi, \pi, t)$$
$$= \frac{\partial v_1}{\partial t} + \left(\frac{\partial v_1}{\partial \pi}\right)^\top f_0(0, \pi, t) + \left(\frac{\partial v_1}{\partial \pi}\right)^\top [f_0(\xi, \pi, t) - f_0(0, \pi, t)]$$
$$\leq -\gamma_3 \|\pi\|^2 + \gamma_4 k_1 \|\xi\| \|\pi\|$$
$$\leq -\gamma_3 \|\pi\|^2 + \gamma_4 k_1 k_2 \|\pi\|.$$

Therefore, $\dot{v}_1 \leq 0$ if $\|\pi\| \geq \gamma_4 k_1 k_2 / \gamma_3$. This ensures boundedness of π uniformly.

4. Indirect adaptive control

The on-line learning abilities of neural networks are considered here to approximate the time-varying dynamics of the non-linear system. In particular, the linear in the parameter radial basis function networks are in the form of

$$\hat{\alpha}(x, t) = \theta_\alpha^\top(t)\phi_\alpha(x) \tag{21}$$

$$\hat{\beta}(x, t) = \theta_\beta^\top(t)\phi_\beta(x) \tag{22}$$

where the parameter vectors $\theta_\alpha(t)$ and $\theta_\beta(t)$ are updated on-line and are assumed to be defined within the compact parameter sets Ω_α and Ω_β, respectively. In addition, we define the subspace $S_x \subseteq \Re^n$ as the space through which the state trajectory may travel under closed-loop control (we are making no a priori assumptions here about the size of S_x). We also define the actual system as

$$\alpha(x, t) = \theta_\alpha^{*\top}(t)\phi_\alpha(x) + \omega_\alpha(x, t) \tag{23}$$

$$\beta(x, t) = \theta_\beta^{*\top}(t)\phi_\beta(x) + \omega_\beta(x, t) \tag{24}$$

where

$$\theta_\alpha^*(t) = \arg \min_{\theta_\alpha \in \Omega_\alpha} \left(\sup_{x \in S_x} |\theta_\alpha^\top \phi_\alpha(x) - \alpha(x, t)| \right) \tag{25}$$

$$\theta_\beta^*(t) = \arg \min_{\theta_\beta \in \Omega_\beta} \left(\sup_{x \in S_x} |\theta_\beta^\top \phi_\beta(x) - \beta(x, t)| \right) \tag{26}$$

are the optimal time-varying parameters and $\omega_\alpha(x, t)$ and $\omega_\beta(x, t)$ are approximation errors which arise when $\alpha(x, t)$ and $\beta(x, t)$ are represented by finite size approximators. We assume that

$$|\omega_\alpha(x, t)| \leq W_\alpha(x) \tag{27}$$

$$|\omega_\beta(x, t)| \leq W_\beta(x) \tag{28}$$

where $W_\alpha(x)$ and $W_\beta(x)$ are known state-dependent time-invariant bounds on the errors in representing the actual system with approximators. We also define parameter errors to be

$$\tilde{\theta}_\alpha(t) = \theta_\alpha(t) - \theta_\alpha^*(t) \tag{29}$$

$$\tilde{\theta}_\beta(t) = \theta_\beta(t) - \theta_\beta^*(t). \tag{30}$$

We view adaptive control to be a tracking problem, that is, to design a control system which will cause the output $y(t)$ and its derivatives $\dot{y}(t), \ldots, y^{(d)}(t)$ to track a desired reference trajectory $y_m(t)$ and its derivatives $\dot{y}_m(t), \ldots, y_m^{(d)}(t)$, respectively, which we assume to be bounded. The reference trajectory may be defined by a reference signal whose first d derivatives are measurable, or by any reference input $r(t)$ passing through a reference model, with relative degree equal to or greater than d. In particular, a linear reference model may be

$$\frac{Y_m(s)}{R(s)} = \frac{q(s)}{p(s)} = \frac{q_0}{s^d + p_{d-1}s^{d-1} + \cdots + p_0} \tag{31}$$

where $p(s)$ is the pole polynomial with stable roots.

The indirect adaptive control law

$$u = u_{ce} + u_{si} \tag{32}$$

comprises a 'certainty equivalence' control term u_{ce} (based on the approximated system dynamics) and a 'sliding mode' control term u_{si} (to compensate for approximation errors). Let the tracking error be $e(t) = y_m(t) - y(t)$ and a measure of the tracking error be $e_s(t) = e^{(d-1)}(t) + k_{d-2}e^{(d-2)}(t) + \cdots + k_1\dot{e}(t) + k_0 e(t)$, that is, in the frequency domain, $e_s(s) = L(s)e(s)$ with $L(s) = s^{(d-1)} + k_{d-2}\, s^{(d-2)} + \cdots + k_1 s + k_0$ whose roots are chosen to be in the (open) left half plane. Also, for convenience below we let $\bar{e}_s(t) = \dot{e}_s(t) - e^{(d)}(t)$. Notice that our control goal is to drive $e_s(t) \to 0$ as $t \to \infty$ and the shape of the error dynamics is dictated by the choice of the design parameters in $L(s)$. The certainty equivalence control term is defined as

$$u_{ce} = \frac{1}{\beta_k(t) + \hat{\beta}(x, t)}(-(\alpha_k(t) + \hat{\alpha}(x, t)) + \nu(t)) \tag{33}$$

where $\beta_k(t) + \hat{\beta}(x, t)$ is bounded away from zero (which will be ensured later using projection) so that u_{ce} is well defined, and

$$\nu(t) = y_m^{(d)} + \eta e_s + \bar{e}_s \tag{34}$$

with $\eta > 0$ as a design parameter. Consider the update laws

$$\dot{\theta}_\alpha(t) = -Q_\alpha^{-1}\phi_\alpha(x)e_s \tag{35}$$

$$\dot{\theta}_\beta(t) = -Q_\beta^{-1}\phi_\beta(x)e_s u_{ce} \tag{36}$$

where Q_α and Q_β are positive definite and diagonal and serve as adaptation gains for the parameter updates. Note that the above adaptation laws do not guarantee that $\theta_\alpha \in \Omega_\alpha$ and $\theta_\beta \in \Omega_\beta$ so that we will use a projection method to ensure this, in particular, to make sure that $\beta_k(t)+\hat{\beta}(x,t)\geq\beta_0$. Additionally, the sliding mode control term is defined as

$$u_{si}=\frac{(W_\alpha(x)+W_\beta(x)|u_{ce}|)}{\beta_0}\mathrm{sgn}(e_s). \quad (37)$$

To develop a stable adaptive controller for non-linear time-varying systems, some assumptions about the form of the model and other technical conditions are necessary. Here we give several possible assumptions on the characteristics of time-varying dynamics and summarize the stability results in the following theorems.

4.1. *Bounded parameter variations*

Similar to linear time-varying systems (Tsakalis and Ioannou 1993), a common assumption with respect to non-linear time-varying systems is to assume the boundedness of parameter variations, that is, we assume $\dot{\theta}^*_\alpha$ and $\dot{\theta}^*_\beta$ are bounded.

Theorem 1: *Consider the non-linear time-varying system* (6) *and* (7) *with strong relative degree* d. *Assume that* (i) $\alpha_k(t)$ *and* $\beta_k(t)$ *in* (16) *are bounded if* x *is bounded*, (ii) $\beta_k(t)+\beta(x,t)\geq\beta_0$ *for some known* $\beta_0>0$, (iii) $|\omega_\alpha(x,t)|\leq W_\alpha(x)$ *and* $|\omega_\beta(x,t)|\leq W_\beta(x)$ *with known* $W_\alpha(x)$ *and* $W_\beta(x)$, (iv) $y_m(t),\dot{y}_m(t),\ldots,y_m^{(d)}(t)$ *are measurable and bounded*, (v) $x(t),y(t),\dot{y}(t),\ldots,y^{(d-1)}(t)$ *are measurable*, (vi) $1\leq d<n$ *with the zero dynamics uniformly exponentially attractive or* $d=n$, *and* (vii) $\dot{\theta}^*_\alpha$ *and* $\dot{\theta}^*_\beta$ *are bounded. Under these conditions there exist indirect adaptive control laws* (*Narendra and Parthasarathy 1990*, *Sanner and Slotine 1992*, *Polycarpou and Mears 1998*) *and update laws* (*Yabuta and Yamada 1992*, *Sadegh 1993*) *such that all internal signals are uniformly bounded and the tracking error* e *is 'small in the mean'.*

Proof: Consider the Lyapunov function candidate

$$V_i(e_s,\theta_\alpha,\theta_\beta,t)=\frac{1}{2}e_s^2+\frac{1}{2}\tilde{\theta}_\alpha^\top Q_\alpha\tilde{\theta}_\alpha+\frac{1}{2}\tilde{\theta}_\beta^\top Q_\beta\tilde{\theta}_\beta \quad (38)$$

which quantifies both errors in tracking and in parameter estimation. It can be easily seen that

$$\frac{1}{2}e_s^2\leq V_i(e_s,\theta_\alpha,\theta_\beta,t)\leq\frac{1}{2}e_s^2+m \quad (39)$$

for some $m>0$ according to the boundedness of $\tilde{\theta}_\alpha$ and $\tilde{\theta}_\beta$. Hence, V_i is positive definite and decrescent. Using vector derivatives and following Spooner and Passino (1996) and Diao and Passino (2001), the time derivative of (38) is

$$\dot{V}_i=e_s\dot{e}_s+\tilde{\theta}_\alpha^\top Q_\alpha\dot{\tilde{\theta}}_\alpha+\tilde{\theta}_\beta^\top Q_\beta\dot{\tilde{\theta}}_\beta.$$

Note that $\bar{e}_s(t)=\dot{e}_s(t)-e^{(d)}(t)$ and the dth derivative of the output error is $e^{(d)}=y_m^{(d)}-y^{(d)}$ so that

$$\dot{e}_s(t)=\bar{e}_s(t)+y_m^{(d)}-y^{(d)}$$

and from (16), (32), (33) and (34)

$$\begin{aligned}\dot{e}_s(t)&=\bar{e}_s(t)+[\nu(t)-\eta e_s-\bar{e}_s]-[(\alpha_k(t)+\alpha(x,t))\\&\quad+(\beta_k(t)+\beta(x,t))(u_{ce}+u_{si})]\\&=-\eta e_s+[\nu(t)-\alpha_k(t)-(\beta_k(t)+\hat{\beta}(x,t))u_{ce}]\\&\quad-\alpha(x,t)+(\hat{\beta}(x,t)-\beta(x,t))u_{ce}-(\beta_k(t)+\beta(x,t))u_{si}\\&=-\eta e_s+[\nu(t)-\alpha_k(t)+(\alpha_k(t)+\hat{\alpha}(x,t))-\nu(t)]\\&\quad-\alpha(x,t)+(\hat{\beta}(x,t)-\beta(x,t))u_{ce}-(\beta_k(t)+\beta(x,t))u_{si}\\&=-\eta e_s+(\hat{\alpha}(x,t)-\alpha(x,t))+(\hat{\beta}(x,t)\\&\quad-\beta(x,t))u_{ce}-(\beta_k(t)+\beta(x,t))u_{si}\end{aligned}$$

also from (21) and (22), (23) and (24) and (29) and (30) we have

$$\begin{aligned}\dot{e}_s&=-\eta e_s+(\tilde{\theta}_\alpha^\top\phi_\alpha(x)-\omega_\alpha(x,t))+(\tilde{\theta}_\beta^\top\phi_\beta(x)\\&\quad-\omega_\beta(x,t))u_{ce}-(\beta_k(t)+\beta(x,t))u_{si}.\end{aligned}$$

Substitute the above equation into (39) and substitute (35) and (36) into (39)

$$\begin{aligned}\dot{V}_i&=e_s[-\eta e_s+(\tilde{\theta}_\alpha^\top\phi_\alpha(x)-\omega_\alpha(x,t))+(\tilde{\theta}_\beta^\top\phi_\beta(x)\\&\quad-\omega_\beta(x,t))u_{ce}-(\beta_k(t)+\beta(x,t))u_{si}]\\&\quad+\tilde{\theta}_\alpha^\top Q_\alpha[-Q_\alpha^{-1}\phi_\alpha(x)e_s]+\tilde{\theta}_\beta^\top Q_\beta[-Q_\beta^{-1}\phi_\beta(x)e_su_{ce}]\\&\quad-\left[\tilde{\theta}_\alpha^\top Q_\alpha\dot{\theta}^*_\alpha+\tilde{\theta}_\beta^\top Q_\beta\dot{\theta}^*_\beta\right]\\&=-\eta e_s^2-(\omega_\alpha(x,t)+\omega_\beta(x,t)u_{ce})e_s\\&\quad-(\beta_k(t)+\beta(x,t))u_{si}e_s-\left[\tilde{\theta}_\alpha^\top Q_\alpha\dot{\theta}^*_\alpha+\tilde{\theta}_\beta^\top Q_\beta\dot{\theta}^*_\beta\right].\end{aligned}$$

Note that we did not consider projection modification to the update laws above. Clearly, since $\theta^*_\alpha\in\Omega_\alpha$ and $\theta^*_\beta\in\Omega_\beta$, when the projection is in effect it always results in smaller parameter errors that will decrease V_i so that

$$\begin{aligned}\dot{V}_i&\leq-\eta e_s^2-(\omega_\alpha(x,t)+\omega_\beta(x,t)u_{ce})e_s-(\beta_k(t)\\&\quad+\beta(x,t))u_{si}e_s-\left[\tilde{\theta}_\alpha^\top Q_\alpha\dot{\theta}^*_\alpha+\tilde{\theta}_\beta^\top Q_\beta\dot{\theta}^*_\beta\right].\end{aligned}$$

Note that the combined term $-[\tilde{\theta}_\alpha^\top Q_\alpha\dot{\theta}^*_\alpha+\tilde{\theta}_\beta^\top Q_\beta\dot{\theta}^*_\beta]$ reflects the effect of time-varying parameters. Since $\tilde{\theta}_\alpha$ and $\tilde{\theta}_\beta$ are bounded according to parameter projection, Q_α and Q_β are constant matrices, and $\dot{\theta}^*_\alpha$ and $\dot{\theta}^*_\beta$ are bounded under the above assumption, we have

$$-\left[\tilde{\theta}_\alpha^\top Q_\alpha\dot{\theta}^*_\alpha+\tilde{\theta}_\beta^\top Q_\beta\dot{\theta}^*_\beta\right]\leq W_\gamma \quad (40)$$

for some $W_\gamma>0$ (which is a constant measuring the boundedness of parameter variations but does not need to be known) so that

$$\begin{aligned}\dot{V}_i&\leq-\eta e_s^2-(\omega_\alpha(x,t)+\omega_\beta(x,t)u_{ce})e_s\\&\quad-(\beta_k(t)+\beta(x,t))u_{si}e_s+W_\gamma.\end{aligned}$$

Substitute (37) into the above equation and also note that $-(\omega_\alpha(x,t)+\omega_\beta(x,t)u_{ce})e_s \le (|\omega_\alpha(x,t)| + |\omega_\beta(x,t)u_{ce}|)\ |e_s| \le (W_\alpha(x)+W_\beta(x)|u_{ce}|)|e_s|$, $\beta_k(t) + \beta(x,t) \ge \beta_0$, and $|e_s| = e_s\text{sgn}(e_s)$ (except at $e_s = 0$)

$$\begin{aligned}\dot{V}_i &\le -\eta e_s^2 + (W_\alpha(x)+W_\beta(x)|u_{ce}|)|e_s| - (\beta_k(t) \\ &\quad + \beta(x,t))\left(\frac{(W_\alpha(x)+W_\beta(x)|u_{ce}|)}{\beta_0}\text{sgn}(e_s)\right)e_s + W_\gamma \\ &\le -\eta e_s^2 + (W_\alpha(x)+W_\beta(x)|u_{ce}|)|e_s| \\ &\quad - \frac{(\beta_k(t)+\beta(x,t))}{\beta_0}(W_\alpha(x)+W_\beta(x)|u_{ce}|)|e_s| + W_\gamma \\ &\le -\eta e_s^2 + W_\gamma.\end{aligned}$$

Thus, $\dot{V}_i$ is negative definite for $|e_s| \ge \sqrt{W_\gamma/\eta}$, that is, the measure of the tracking error e_s is uniformly bounded. As $e_s(s) = L(s)e(s)$ and $L(s)$ is a stable function with the degree of $d-1$, we know that the tracking error and its derivatives $e, \dot{e}, \dots, e^{(d-1)}$ are uniformly bounded. Since the reference trajectory y_m and its derivatives $\dot{y}_m, \dots, y_m^{(d-1)}$ are assumed to be bounded, the system output y and its derivatives $\dot{y}, \dots, y^{(d-1)}$ are bounded. Hence, ξ is uniformly bounded and thus x is uniformly bounded. Besides, the fact that $\dot{V}_i$ is negative definite also implies that parameter estimations θ_α and θ_β are uniformly bounded (noting (29) and (30) and the boundedness of θ_α^* and θ_β^*.). Therefore, the boundedness of $\hat{\alpha}(x,t)$, $\hat{\beta}(x,t)$, $\alpha_k(t)$, and $\beta_k(t)$ assures that u_{ce} and u_{si} and hence u are uniformly bounded.

$\dot{V}_i \le 0$ for $|e_s| \ge \sqrt{W_\gamma/\eta}$ also assures that

$$\begin{aligned}\int_t^{t+T} e_s^2\,\mathrm{d}t &\le -\frac{1}{\eta}\int_t^{t+T}\dot{V}_i\,\mathrm{d}t + \frac{1}{\eta}\int_t^{t+T} W_\gamma\,\mathrm{d}t \\ &= \frac{1}{\eta}(V_i(t) - V_i(t+T)) + \frac{W_\gamma}{\eta}T\end{aligned}$$

that is, the tracking error e_s is 'small in the mean'. □

Remark: Note that η is a design parameter and that by choosing it large we can obtain smaller mean in the above result.

4.2. *Bounded parameter rate of change*

Note that for the time-varying systems, although we can guarantee that all internal signals are bounded uniformly, under the above assumption about the boundedness of parameter variations, we can only show that the tracking error e_s is small in the mean and $e_s \in L_2$ may not be established. In order to obtain the uniform asymptotic stability of the output we may need a stronger assumption such as

$$|\dot{\theta}_{\alpha,i}^*| \le k|e_s| \tag{41}$$

$$|\dot{\theta}_{\beta,j}^*| \le k|e_s| \tag{42}$$

where $\dot{\theta}_{\alpha,i}^*$ and $\dot{\theta}_{\beta,j}^*$ are components of the vectors of $\dot{\theta}_\alpha^*$ and $\dot{\theta}_\beta^*$, respectively, and k is a positive constant. This assumption is reasonable because the tracking error is usually large if the plant parameters vary fast (on the other hand the condition may be difficult to verify in specific applications). Under the above assumption we get

$$-\left[\tilde{\theta}_\alpha^\top Q_\alpha \dot{\theta}_\alpha^* + \tilde{\theta}_\beta^\top Q_\beta \dot{\theta}_\beta^*\right] \le W_\gamma |e_s| \tag{43}$$

for some known constant $W_\gamma > 0$ indicating the bounds of parameter rate of change with respect to $|e_s|$. In addition, we redesign the sliding mode control term as

$$u_{si} = \frac{(W_\alpha(x)+W_\beta(x)|u_{ce}|)}{\beta_0}\text{sgn}(e_s) + \frac{W_\gamma}{\beta_0}\text{sgn}(e_s) \tag{44}$$

where the first term is the same as (37) to compensate for approximation errors and the second term is added to compensate for the time-varying dynamics.

Theorem 2: *Consider the non-linear time-varying system* (6) *and* (7) *with strong relative degree d. Assume that* (i) $\alpha_k(t)$ *and* $\beta_k(t)$ *in* (16) *are bounded if x is bounded,* (ii) $\beta_k(t) + \beta(x,t) \ge \beta_0$ *for some known* $\beta_0 > 0$, (iii) $|\omega_\alpha(x,t)| \le W_\alpha(x)$ *and* $|\omega_\beta(x,t)| \le W_\beta(x)$ *with known* $W_\alpha(x)$ *and* $W_\beta(x)$, (iv) $y_m(t), \dot{y}_m(t), \dots, y_m^{(d)}(t)$ *are measurable and bounded,* (v) $x(t), y(t), \dot{y}(t), \dots, y^{(d-1)}(t)$ *are measurable,* (vi) $1 \le d < n$ *with the zero dynamics uniformly exponentially attractive or* $d = n$, *and* (vii) $|\dot{\theta}_{\alpha,i}^*| \le k|e_s|$ *and* $|\dot{\theta}_{\beta,j}^*| \le k|e_s|$. *Under these conditions there exist indirect adaptive control laws* (32), (33), (44) *and update laws* (35) *and* (36) *such that all internal signals are uniformly bounded and the tracking error e is uniformly asymptotically stable.*

Proof: Consider the Lyapunov function candidate

$$V_i(e_s, \theta_\alpha, \theta_\beta, t) = \frac{1}{2}e_s^2 + \frac{1}{2}\tilde{\theta}_\alpha^\top Q_\alpha \tilde{\theta}_\alpha + \frac{1}{2}\tilde{\theta}_\beta^\top Q_\beta \tilde{\theta}_\beta \tag{45}$$

and its time derivative

$$\dot{V}_i = \left[e_s\dot{e}_s + \tilde{\theta}_\alpha^\top Q_\alpha \dot{\theta}_\alpha + \tilde{\theta}_\beta^\top Q_\beta \dot{\theta}_\beta\right] - \left[\tilde{\theta}_\alpha^\top Q_\alpha \dot{\theta}_\alpha^* + \tilde{\theta}_\beta^\top Q_\beta \dot{\theta}_\beta^*\right]. \tag{46}$$

Use the same derivation procedure as in the proof of Theorem 1 to get

$$\begin{aligned}\dot{V}_i &\le -\eta e_s^2 - (\omega_\alpha(x,t)+\omega_\beta(x,t)u_{ce})e_s - (\beta_k(t) \\ &\quad + \beta(x,t))u_{si}e_s - \left[\tilde{\theta}_\alpha^\top Q_\alpha \dot{\theta}_\alpha^* + \tilde{\theta}_\beta^\top Q_\beta \dot{\theta}_\beta^*\right].\end{aligned}$$

Substitute (43) and (44) into the above equation and also note that $-(\omega_\alpha(x,t)+\omega_\beta(x,t)u_{ce})e_s \le (|\omega_\alpha(x,t)| + |\omega_\beta(x,t)u_{ce}|)|e_s| \le (W_\alpha(x)+W_\beta(x)|u_{ce}|)|e_s|$ and $\beta_k(t) + \beta(x,t) \ge \beta_0$.

$$\begin{aligned}\dot{V}_i &\leq -\eta e_s^2 + (W_\alpha(x) + W_\beta(x)|u_{ce}|)|e_s| - (\beta_k(t) + \beta(x,t)) \\ &\quad \times \left(\frac{(W_\alpha(x) + W_\beta(x)|u_{ce}|)}{\beta_0}\operatorname{sgn}(e_s) + \frac{W_\gamma}{\beta_0}\operatorname{sgn}(e_s)\right)e_s \\ &\quad + W_\gamma|e_s| \\ &\leq -\eta e_s^2 + (W_\alpha(x) + W_\beta(x)|u_{ce}|)|e_s| \\ &\quad - \frac{(\beta_k(t) + \beta(x,t))}{\beta_0}(W_\alpha(x) + W_\beta(x)|u_{ce}|)|e_s| \\ &\quad - \frac{(\beta_k(t) + \beta(x,t))}{\beta_0}W_\gamma|e_s| + W_\gamma|e_s| \\ &\leq -\eta e_s^2.\end{aligned}$$

Thus, $\dot{V}_i$ is negative definite, and we can obtain the uniform boundedness of all internal signals using a similar analysis as in the proof of Theorem 1. Furthermore, noting that

$$\int_0^\infty \eta e_s^2\,\mathrm{d}t \leq -\int_0^\infty \dot{V}_i\,\mathrm{d}t = V_i(0) - V_i(\infty) \tag{47}$$

this establishes that $e_s \in L_2$ ($L_2 = \{z(t): \int_0^\infty z^2(t) < \infty\}$) since $V_i(0)$ and $V_i(\infty)$ are bounded. Since e_s and $\dot{e}_s$ are bounded and $e_s \in L_2$, by Barbalat's Lemma we have uniform asymptotic stability of e_s, which implies uniform asymptotic stability of the tracking error e (i.e., $\lim_{t\to\infty} e = 0$). □

Alternatively, we may assume that

$$|\dot{\theta}^*_{\alpha,i}| \leq r\mathrm{e}^{-kt} \tag{48}$$

$$|\dot{\theta}^*_{\beta,j}| \leq r\mathrm{e}^{-kt} \tag{49}$$

where $\dot{\theta}^*_{\alpha,i}$ and $\dot{\theta}^*_{\beta,j}$ are components of the vectors of $\dot{\theta}^*_\alpha$ and $\dot{\theta}^*_\beta$, respectively, and $r>0$ and $k>0$ are some constants. This assumption may be used to represent a class of time-varying systems, such as the systems with incipient faults or jump-like faults, where the time-varying effects fade as the time goes to infinity. Again, although this is a reasonable assumption (e.g., see Goodwin *et al.* (1984) for linear time-varying systems), it may be difficult to verify it in practice.

Theorem 3: *Consider the non-linear time-varying system* (6) *and* (7) *with strong relative degree d. Assume that* (i) $\alpha_k(t)$ *and* $\beta_k(t)$ *in* (16) *are bounded if x is bounded*, (ii) $\beta_k(t) + \beta(x,t) \geq \beta_0$ *for some known* $\beta_0 > 0$, (iii) $|\omega_\alpha(x,t)| \leq W_\alpha(x)$ *and* $|\omega_\beta(x,t)| \leq W_\beta(x)$ *with known* $W_\alpha(x)$ *and* $W_\beta(x)$, (iv) $y_m(t), \dot{y}_m(t), \ldots, y_m^{(d)}(t)$ *are measurable and bounded*, (v) $x(t), y(t), \dot{y}(t), \ldots, y^{(d-1)}(t)$ *are measurable*, (vi) $1 \leq d < n$ *with the zero dynamics uniformly exponentially attractive or* $d = n$, *and* (vii) $|\dot{\theta}^*_{\alpha,i}| \leq r\,\mathrm{e}^{-kt}$ *and* $|\dot{\theta}^*_{\beta,j}| \leq r\,\mathrm{e}^{-kt}$. *Under these conditions there exist indirect adaptive control laws* (32), (33) *and* (37) *and update laws* (35) *and* (36) *such that all internal signals are uniformly bounded and the tracking error e is uniformly asymptotically stable.*

Proof: Under the above assumption we obtain

$$-\left[\tilde{\theta}_\alpha^\top Q_\alpha \dot{\theta}^*_\alpha + \tilde{\theta}_\beta^\top Q_\beta \dot{\theta}^*_\beta\right] \leq W_\gamma \mathrm{e}^{-kt}. \tag{50}$$

Similar to the stability analysis procedure in Theorem 1 we get

$$\begin{aligned}\dot{V}_i &\leq -\eta e_s^2 - \left[\tilde{\theta}_\alpha^\top Q_\alpha \dot{\theta}^*_\alpha + \tilde{\theta}_\beta^\top Q_\beta \dot{\theta}^*_\beta\right] \\ &\leq -\eta e_s^2 + W_\gamma\,\mathrm{e}^{-kt}.\end{aligned}$$

Therefore

$$\begin{aligned}\int_0^\infty \eta e_s^2\mathrm{d}t &\leq -\int_0^\infty \dot{V}_i\mathrm{d}t + W_\gamma\int_0^\infty \mathrm{e}^{-kt}\mathrm{d}t \\ &= V_i(0) - V_i(\infty) + \frac{W_\gamma}{k}\end{aligned} \tag{51}$$

which establishes that $e_s \in L_2$. Thus, by Barbalat's Lemma we have uniform asymptotic stability of the tracking error e. □

Remark: Our work on stable adaptive control of feedback linearizable time-varying non-linear systems is motivated with the fault-tolerant engine control problem. Most existing studies on fault diagnosis and fault-tolerant control have relied on a linear nominal model of the plant. However, in practical situations plants are non-linear and the faults often force plants away from local behaviours that are locally linear into a non-linear operating region. Furthermore, the existing work in the literature mainly considers fault-tolerant control in the context of time-invariant systems as if a fault has already occurred, while the reality is that both incipient and abrupt faults are naturally time-varying phenomena. Therefore, existing work on using on-line approximation approaches for non-linear time-invariant systems (Spooner *et al.* 2001) and using robust adaptive control for linear time-varying systems (Tsakalis and Ioannou 1993) are not applicable. One method to address the problem of adaptive control for non-linear time-varying systems is to use the backstepping design methodology for the class of systems in the strict feedback form (Marino and Tomei 1997, 1998, Ordonez and Passino 2000 b, c). However, the actual plant may not fit this form (and, in particular, the engine does not). Here we consider a more general class of non-linear time-varying systems, which is input–output feedback linearizable. This class of systems is large enough so that it is not only of theoretical interest but also of practical applicability (e.g. to our fault-tolerant engine control problem). The adaptive control law is designed to generalize the existing robust adaptive fuzzy/neural control method (Spooner and Passino 1996, Diao and Passino 2001) to the time-varying cases by taking into account uncertain time-varying parameters (with known bounds) (Marino and Tomei 1998). Furthermore, a stronger stability result, uniform asymptotic stability of the output, can be obtained by assuming boundedness of parameter rate of change.

5. Direct adaptive control

In addition to the assumptions we made in the indirect adaptive control case, we require $\beta_k(t) = \alpha_k(t) = 0$ for all $t \geq 0$, and that there exist positive constants β_0 and β_1 such that $0 < \beta_0 \leq \beta(x,t) \leq \beta_1$. Also, we assume that we can specify some function $B(x) \geq 0$ such that

$$|\dot{\beta}(x,t)| = \left| \frac{\partial \beta}{\partial t} + \left(\frac{\partial \beta}{\partial x} \right)^\top \dot{x} \right| \leq B(x)$$

for all $x \in S_x$. We know that there exists some ideal controller

$$u^* = \frac{1}{\beta(x,t)} (-\alpha(x,t) + \nu(t)) \tag{52}$$

where $\nu(t)$ is defined the same as that in the indirect adaptive control case (34). We also define the ideal controller (in the form of the ideal approximator) as

$$u^* = \theta_u^{*\top}(t) \phi_u(x,\nu) + u_k(t) + \omega_u(x,\nu,t) \tag{53}$$

where u_k is a known part of the controller (e.g. one that was designed for the nominal system), the parameters

$$\theta_u^*(t) = \arg \min_{\theta_u \in \Omega_u} \left(\sup_{x \in S_x, \, \nu \in S_\nu} |\theta_u^\top \phi_u(x,\nu) - (u^* - u_k)| \right) \tag{54}$$

are the optimal time-varying parameters of the approximator, and $\omega_u(x,\nu,t)$ is the approximation error. We assume that $|\omega_u(x,\nu,t)| \leq W_u(x,\nu)$, where $W_u(x,\nu)$ is a known bound on the error in representing the ideal controller. The approximation of this ideal controller can be represented by

$$\hat{u} = \theta_u^\top \phi_u(x,\nu) + u_k(t) \tag{55}$$

where the parameter vector $\theta_u(t)$ is updated on-line and the parameter error is

$$\tilde{\theta}_u(t) = \theta_u(t) - \theta_u^*(t). \tag{56}$$

Consider the direct adaptive control law

$$u = \hat{u} + u_{sd} \tag{57}$$

which is the sum of the approximation to the ideal control law $\hat{u}$ and a sliding mode control term

$$u_{sd} = \left(\frac{B(x)|e_s|}{2\beta_0^2} + W_u(x,\nu) \right) \operatorname{sgn}(e_s) \tag{58}$$

and we use the update law

$$\dot{\theta}_u(t) = Q_u^{-1} \phi_u(x,\nu) e_s(t) \tag{59}$$

where Q_u is positive definite and diagonal. We also use a projection method to ensure that $\theta_u \in \Omega_u$.

5.1. *Bounded controller parameters*

Analogous to indirect adaptive control, a reasonable assumption regarding to stable direct adaptive control of non-linear time-varying systems is to assume the boundedness of controller parameter variations $\dot{\theta}_u^*$, which may result from the bounded time-varying characteristics of the plants.

Theorem 4: *Consider the non-linear time-varying system* (6) *and* (7) *with strong relative degree d. Assume that* (i) $0 < \beta_0 \leq \beta(x,t) \leq \beta_1$ *for some known positive constants β_0 and β_1,* (ii) $|\dot{\beta}(x,t)| \leq B(x)$ *for some known function $B(x) \geq 0$,* (iii) $|\omega_u(x,\nu,t)| \leq W_u(x,\nu)$ *with known $W_u(x,\nu)$,* (iv) $y_m(t), \dot{y}_m(t), \ldots, y_m^{(d)}(t)$ *are measurable and bounded,* (v) $x(t), y(t), \dot{y}(t), \ldots, y^{(d-1)}(t)$ *are measurable,* (vi) $1 \leq d < n$ *with the zero dynamics uniformly exponentially attractive or $d = n$, and* (vii) $\dot{\theta}_u^*$ *is bounded. Under these conditions there exist direct adaptive control laws* (57), (55) *and* (58) *and update laws* (59) *such that all internal signals are uniformly bounded and the tracking error e is 'small in the mean'.*

Proof: Analogous to Spooner and Passino (1996), consider the following Lyapunov function candidate

$$V_d = \frac{1}{2\beta(x,t)} e_s^2 + \frac{1}{2} \tilde{\theta}_u^\top Q_u \tilde{\theta}_u. \tag{60}$$

It can be easily seen that

$$\frac{1}{2\beta_1} e_s^2 \leq V_d \leq \frac{1}{2\beta_0} e_s^2 + m$$

for some $m > 0$ according to the boundedness of $\beta(x,t)$ and $\tilde{\theta}_u$. Hence, V_d is positive definite and decrescent. Take the time derivative

$$\dot{V}_d = \frac{e_s}{\beta(x,t)} \dot{e}_s - \frac{\dot{\beta}(x,t) e_s^2}{2\beta^2(x,t)} + \tilde{\theta}_u^\top Q_u \dot{\tilde{\theta}}_u. \tag{61}$$

Note that $\bar{e}_s(t) = \dot{e}_s(t) - e^{(d)}(t)$ and the dth derivative of the output error is $e^{(d)} = y_m^{(d)} - y^{(d)}$ so that

$$\dot{e}_s(t) = \bar{e}_s(t) + y_m^{(d)} - y^{(d)}$$

and from (16), (34), (52) and (57) and by assuming $\alpha_k(t) = \beta_k(t) = 0$

$$\begin{aligned} \dot{e}_s(t) &= \bar{e}_s(t) + [\nu(t) - \eta e_s - \bar{e}_s] \\ &\quad - [(\alpha_k(t) + \alpha(x,t)) + (\beta_k(t) + \beta(x,t))(\hat{u} + u_{sd})] \\ &= -\eta e_s + [\nu(t) - \alpha(x,t) - \beta(x,t) u^*] \\ &\quad - \beta(x,t)(\hat{u} - u^*) - \beta(x,t) u_{sd} \\ &= -\eta e_s - \beta(x,t)(\hat{u} - u^*) - \beta(x,t) u_{sd} \end{aligned}$$

also from (53), (55) and (56) we have

$$\dot{e}_s(t) = -\eta e_s - \beta(x,t)(\tilde{\theta}_u^\top \phi_u(x,\nu) - \omega_u(x,\nu,t)) - \beta(x,t) u_{sd}. \tag{62}$$

Substitute the above equation into (61) and substitute (59) into (61)

$$\dot{V}_d = \frac{e_s}{\beta(x,t)}[-\eta e_s - \beta(x,t)(\tilde{\theta}_u^\top \phi_u(x,\nu) - \omega_u(x,\nu,t)) - \beta(x,t)u_{sd}] - \frac{\dot{\beta}(x,t)e_s^2}{2\beta^2(x,t)} + \tilde{\theta}_u^\top Q_u[Q_u^{-1}\phi_u(x,\nu)e_s(t)] - \tilde{\theta}_u^\top Q_u \dot{\theta}_u^*$$

$$= -\frac{\eta e_s^2}{\beta(x,t)} - \left(\tilde{\theta}_u^\top \phi_u - \omega_u(x,\nu,t) + \frac{\dot{\beta}(x,t)e_s}{2\beta^2(x,t)} - \tilde{\theta}_u^\top \phi_u\right)e_s - e_s u_{sd} - \tilde{\theta}_u^\top Q_u \dot{\theta}_u^*$$

$$= -\frac{\eta e_s^2}{\beta(x,t)} - \left(\frac{\dot{\beta}(x,t)e_s}{2\beta^2(x,t)} - \omega_u(x,\nu,t)\right)e_s - e_s u_{sd} - \tilde{\theta}_u^\top Q_u \dot{\theta}_u^*.$$

After we consider the projection modification to the update law we have

$$\dot{V}_d \le -\frac{\eta e_s^2}{\beta(x,t)} - \left(\frac{\dot{\beta}(x,t)e_s}{2\beta^2(x,t)} - \omega_u(x,\nu,t)\right)e_s - e_s u_{sd} - \tilde{\theta}_u^\top Q_u \dot{\theta}_u^*. \tag{63}$$

Note that the term $-\tilde{\theta}_u^\top Q_u \dot{\theta}_u^*$ reflects the effect of time-varying parameters. Since $\tilde{\theta}_u$ is bounded according to parameter projection, Q_u is a constant matrix, and $\dot{\theta}_u^*$ is bounded under the above assumption, we have

$$-\tilde{\theta}_u^\top Q_u \dot{\theta}_u^* \le W_\gamma \tag{64}$$

for some $W_\gamma > 0$ indicating the boundedness of parameter variations (but which do not need to be known) so that

$$\dot{V}_d \le -\frac{\eta e_s^2}{\beta(x,t)} - \left(\frac{\dot{\beta}(x,t)e_s}{2\beta^2(x,t)} - \omega_u(x,\nu,t)\right)e_s - e_s u_{sd} + W_\gamma. \tag{65}$$

Substitute (58) into the above equation and note that

$$-\left(\frac{\dot{\beta}(x,t)e_s}{2\beta^2(x,t)} - \omega_u(x,\nu,t)\right)e_s \le \left(\frac{|\dot{\beta}(x,t)||e_s|}{2\beta^2(x,t)} + |\omega_u(x,\nu,t)|\right)|e_s| \le \left(\frac{B(x)|e_s|}{2\beta_0^2} + W_u(x,\nu)\right)|e_s|$$

and $0 < \beta_0 \le \beta(x) \le \beta_1$ so that we have

$$\dot{V}_d \le -\frac{\eta e_s^2}{\beta_1} + W_\gamma \tag{66}$$

Thus, $\dot{V}_d$ is negative definite for $|e_s| \ge \sqrt{\beta_1 W_\gamma/\eta}$. Similar to Theorem 1, we may find that all internal signals are uniformly bounded and the tracking error e is 'small in the mean'. □

Remark: To get the mean smaller we could tune the controller design parameter by having a larger η.

5.2. *Bounded rate of change of controller parameters*

Again, analogous to the assumption made in § 4.2, in order to obtain the uniform asymptotic stability of the output we assume that

$$|\dot{\theta}_{u,i}^*| \le k|e_s| \tag{67}$$

where $\dot{\theta}_{u,i}^*$ are components of the vectors of $\dot{\theta}_u^*$ and k is a positive constant, since the tracking error is usually large if the controller parameters vary fast.

Theorem 5: *Consider the non-linear time-varying system* (6) *and* (7) *with strong relative degree* d. *Assume that* (i) $0 < \beta_0 \le \beta(x,t) \le \beta_1$ *for some known positive constants* β_0 *and* β_1, (ii) $|\dot{\beta}(x,t)| \le B(x)$ *for some known function* $B(x) \ge 0$, (iii) $|\omega_u(x,\nu,t)| \le W_u(x,\nu)$ *with known* $W_u(x,\nu)$, (iv) $y_m(t), \dot{y}_m(t), \ldots, y_m^{(d)}(t)$ *are measurable and bounded*, (v) $x(t), y(t), \dot{y}(t), \ldots, y^{(d-1)}(t)$ *are measurable*, (vi) $1 \le d < n$ *with the zero dynamics uniformly exponentially attractive or* $d = n$, *and* (vii) $|\dot{\theta}_{u,i}^*| \le k|e_s|$. *Under these conditions there exist direct adaptive control laws* (55) *and* (57) *and*

$$u_{sd} = \left(\frac{B(x)|e_s|}{2\beta_0^2} + W_u(x,\nu)\right)\mathrm{sgn}(e_s) + W_\gamma \mathrm{sgn}(e_s) \tag{68}$$

for some known constant $W_\gamma > 0$ (*indicating the bounds of controller parameter rate of change*) *and update laws* (59) *such that all internal signals are uniformly bounded and the tracking error* e *is uniformly asymptotically stable.*

Proof: Under the above assumption we get

$$-\tilde{\theta}_u^\top Q_u \dot{\theta}_u^* \le W_\gamma |e_s|. \tag{69}$$

Therefore

$$\dot{V}_d \le -\frac{\eta e_s^2}{\beta(x,t)} - \left(\frac{\dot{\beta}(x,t)e_s}{2\beta^2(x,t)} - \omega_u(x,\nu,t)\right)e_s - e_s u_{sd} + W_\gamma |e_s|. \tag{70}$$

Substituting (68) into the above equation we have

$$\dot{V}_d \le -\frac{\eta e_s^2}{\beta_1}. \tag{71}$$

Thus, $\dot{V}_d$ is negative definite and we may obtain the uniform boundedness of all internal signals and uniform asymptotic stability of the tracking error e similar to the analysis in the proof of Theorem 2. □

Alternatively, we may assume that

$$|\dot{\theta}_{u,i}^*| \le r\,\mathrm{e}^{-kt} \tag{72}$$

where $\dot{\theta}_{u,i}^*$ are components of the vectors of $\dot{\theta}_u^*$, and $r > 0$ and $k > 0$ are some constants. This assumption may be used to represent the situations for a class of time-varying systems, where the time-varying effects fade as the time goes to infinity, so that the corresponding controller parameters also tend to be constant.

Theorem 6: *Consider the non-linear time-varying system* (6) *and* (7) *with strong relative degree d. Assume that* (i) $0 < \beta_0 \le \beta(x, t) \le \beta_1$ *for some known positive constants* β_0 *and* β_1, (ii) $|\dot{\beta}(x, t)| \le B(x)$ *for some known function* $B(x) \ge 0$, (iii) $|\omega_u(x, \nu, t)| \le W_u(x, \nu)$ *with known* $W_u(x, \nu)$, (iv) $y_m(t), \dot{y}_m(t), \dots, y_m^{(d)}(t)$ *are measurable and bounded*, (v) $x(t), y(t), \dot{y}(t), \dots, y^{(d-1)}(t)$ *are measurable*, (vi) $1 \le d < n$ *with the zero dynamics uniformly exponentially attractive or* $d = n$, *and* (vii) $|\dot{\theta}^*_{u,i}| \le re^{-kt}$. *Under these conditions there exist direct adaptive control laws* (55), (57) *and* (58) *and update laws* (59) *such that all internal signals are uniformly bounded and the tracking error e is uniformly asymptotically stable.*

Proof: Under the above assumption we get

$$-\tilde{\theta}_u^\top Q_u \dot{\theta}_u^* \le W_\gamma \,\mathrm{e}^{-kt}. \tag{73}$$

Therefore

$$\begin{aligned}\dot{V}_d \le &-\frac{\eta e_s^2}{\beta(x, t)} - \left(\frac{\dot{\beta}(x, t) e_s}{2\beta^2(x, t)} - \omega_u(x, \nu, t)\right) e_s \\ &- e_s u_{sd} + W_\gamma \,\mathrm{e}^{-kt}.\end{aligned} \tag{74}$$

Substituting (58) into the above equation we have

$$\dot{V}_d \le -\frac{\eta e_s^2}{\beta_1} + W_\gamma \,\mathrm{e}^{-kt} \tag{75}$$

so that

$$\begin{aligned}\int_0^\infty \frac{\eta}{\beta_1} e_s^2 \,\mathrm{d}t &\le -\int_0^\infty \dot{V}_i \,\mathrm{d}t + W_\gamma \int_0^\infty e^{-kt} \,\mathrm{d}t \\ &= V_i(0) - V_i(\infty) + \frac{W_\gamma}{k},\end{aligned} \tag{76}$$

which establishes that $e_s \in L_2$. Thus, by Barbalat's Lemma we have uniform asymptotic stability of e_s. □

Remark: Note that most of the papers (Narendra and Parthasarathy 1990, Polycarpou and Ioannu 1991, Sanner and Slotine 1992, Yabuta and Yamada 1992, Liu and Chen 1993, Sadegh 1993, Chen and Liu 1994, Rovithakis and Christodolou 1994, Su and Steparenko 1994, Wang 1994 a, b, Hsu and Fu 1995, Yeşildirek and Lewis 1995, Chen *et al*. 1996, Farrel 1996, Fabri and Kaderkananathan 1996, Lee and Wang 1996, Lewis *et al*. 1996, Palycarpou 1996, Polycarpou and Mears 1998) deal with indirect adaptive control, whereas very few authors (e.g., Rovithakis and Christodoulou 1995, Spooner and Passino 1996) face the direct approach, because it is not always clear how to construct the control law without knowledge of the system dynamics. Here, we design the direct adaptive control law based on the feedback linearizing law (Spooner and Passino 1996) and then generalize it to the time-varying case. Uniform asymptotic stability of the output has also been obtained by assuming boundedness of rate of change of controller parameters. Compared to indirect adaptive control, direct adaptive control usually shows better transient behavior because it may learn and adapt faster (probably due to the fact that it has fewer parameters to be tuned).

6. Simulation examples: fault-tolerant engine control

To study the effectiveness of the proposed adaptive control methods, we apply them to the component level model simulation of an aircraft jet engine (General Electric XTE46). The General Electric XTE46 engine is a simplified, unclassified version of the original IHPTET engine (Adibhatla and Lewis 1997). The component level engine cycle model of the XTE46 engine is a thermodynamic simulation package, where each engine component is simulated. The CLM executes one pass within the digital control's sampling time, and thermodynamic states are assumed to be in equilibrium after each pass through the simulation. The operating condition of the engine is defined by the altitude (ALT), mach number (XM), difference of temperature (DTAMB), and throttle setting represented by power code (PC). The health of the engine is described by ten 'quality parameters' which include the flows and efficiencies of the fan, the compressor, and turbines. The model has three state variables, including the fan rotor speed (XNL), the core rotor speed (XNH), and the temperature at combustor inlet (TMPC). There are six actuators, but the major control variables are the combustor fuel flow (WF36), the exhaust nozzle area (A8), and the variable area bypass injector area (A16). For simplicity, we assume that the fundamental dynamic characteristics of the CLM can be represented by a single-input single-output system in the form

$$\dot{x} = f(x, u, c, p) \tag{77}$$

$$y = h(x, u, c, p) \tag{78}$$

where $x = [\text{XNL}, \text{XNH}]^\top$ is the state vector, $u = \text{WF36}$ is the input variable, $y = \text{XN2}$ is the output of the engine, $c = [\text{ALT, XM, DTAMB, PC}]^\top$ represents the operating condition of the engine, $p = [\text{ZSW2, SEDM2, ZSW7D, SEDM7D, ZSW27, SEDM27, ZSW41, ZSE41, ZSW49, ZSE49}]^\top$ represents the quality parameter vector, $f(\cdot)$ denotes the unknown function representing the non-linear characteristics of the engine, and $h(\cdot) = \text{XNL}$ because the output variable XN2 is just the measurement of the state variable XNL.

To develop a computational model for the XTE46 engine, we perform non-linear system identification to approximate local engine dynamics (specified by fixed values of operating conditions and quality parameters), followed by interpolating these local models to generate the 'global' model (actually, it is a 'regional' model valid in the 'climb' region) (Diao and Passino 2001, 2004). The general form of the model can be described as

$$\dot{x} = f(x, c, p) + g(x, c, p)u \tag{79}$$

$$y = x_1 \tag{80}$$

where

$$f(x, c, p) = \frac{\sum_{i=1}^{N} \boldsymbol{f}(x, c_i, p_i)\mu_i(c, p)}{\sum_{i=1}^{N} \mu_i(c, p)} \tag{81}$$

$$g(x, c, p) = \frac{\sum_{i=1}^{N} \boldsymbol{g}(x, c_i, p_i)\mu_i(c, p)}{\sum_{i=1}^{N} \mu_i(c, p)} \tag{82}$$

$$\boldsymbol{f}(x, c_i, p_i) = \frac{\sum_{j=1}^{R} [a_{j,0}(c_i, p_i) + a_{j,1}(c_i, p_i)x_1 + a_{j,2}(c_i, p_i)x_2]\underline{\boldsymbol{\mu}}_j(x_1)}{\sum_{j=1}^{R} \underline{\boldsymbol{\mu}}_j(x_1)} \tag{83}$$

$$\boldsymbol{g}(x, c_i, p_i) = \frac{\sum_{j=1}^{R} a_{j,3}(c_i, p_i)\underline{\boldsymbol{\mu}}_j(x_1)}{\sum_{j=1}^{R} \underline{\boldsymbol{\mu}}_j(x_1)} \tag{84}$$

where $u = \mathrm{WF36}$ and $x = [x_1, x_2]^\top = [\widehat{\mathrm{XNL}}, \widehat{\mathrm{XNH}}]^\top$ which is positive since the fan speed and the core speed cannot be negative and $x \in S_x$ (a valid speed region). The value of c is the known operating condition vector, p is an unknown quality parameter vector, and c_i, p_i are specified nodes where we establish node models. Also, $f = [f_1, f_2]^\top$, $g = [g_1, g_2]^\top$, $\boldsymbol{f} = [\boldsymbol{f}_1, \boldsymbol{f}_2]^\top$, and $\boldsymbol{g} = [\boldsymbol{g}_1, \boldsymbol{g}_2]^\top$ are 2×1 function vectors, $a_{j,0}, a_{j,1}, a_{j,2}, a_{j,3}$ are 2×1 vectors of linear parameters of Takagi–Sugeno fuzzy systems, $\mu_i(c, p)$ are membership functions of fuzzy interpolation between different operating conditions and quality parameters, and $\underline{\boldsymbol{\mu}}_j(x_1)$ are membership functions describing the non-linearity with respect to x_1. By inspecting the parameters that result from the identification process we found that $a_{j,3}^1(c_i, p_i) > a_{j,3}^2(c_i, p_i) > 0$ and $a_{j,2}^2(c_i, p_i) < a_{j,2}^1(c_i, p_i) < 0$ for any $i = 1, 2, \ldots, N, j = 1, 2, \ldots, R$. Thus, we know that the "relative degree" of the engine is 1 and the engine zero dynamics are uniformly exponentially attractive (Diao and Passino 2001).

6.1. *Indirect adaptive control*

Consider the engine in the form of

$$\dot{y} = f_1(x, c, p) + g_1(x, c, p)u \tag{85}$$

$$= [\alpha_k(c, p_0, t) + \alpha(x, c, p)] + [\beta_k(c, p_0, t) + \beta(x, c, p)]u \tag{86}$$

where $x(t)$ and $y(t)$ are measurable according to the properties of the component level engine model. By studying dynamics of the developed non-linear model we know that $g_1(x, c, p) > 0.32$ so that we can set $\beta_0 = 0.32$. We use our developed engine model to represent the nominal model dynamics $\alpha_k(c, p_0, t)$ and $\beta_k(c, p_0, t)$ by setting the quality parameters to be the nominal value p_0, and they are bounded if x is bounded since the model is in the form of a Takagi–Sugeno fuzzy system. The unknown dynamics $\alpha(x, c, p)$ and $\beta(x, c, p)$ describe both the model uncertainty caused by nominal model inaccuracy and system changes (time-varying characteristics) due to the fault effects. They will be approximated by two radial basis function networks $\hat{\alpha}$ and $\hat{\beta}$ with 11 receptive field units for each. The inputs to the neural networks include two state variables (XNL and XNH), and the parameters are updated on-line to capture the unknown time-varying dynamics affected by model inaccuracy and faults so that fault tolerance can be achieved. Note that the stable adaptive controller will ensure the stability of x_1, and the uniform exponential attractivity of the engine zero dynamics will ensure the stability of the uncontrollable state x_2. Since the relative degree of the system is 1, the error dynamics are simple ($e_s(t) = e(t)$ and $\bar{e}_s(t) = 0$). As we cannot explicitly know the model uncertainty, the parameters W_α and W_β are treated as design parameters and tuned by trial and error to achieve good control performance. Here, we have $W_\alpha = 0.01$ and $W_\beta = 0.01$. As for the parameter W_γ in (44), since its effect on the sliding mode control term is the same as that of W_α, we just treat it as part of W_α and do not tune it explicitly. In addition, the adaptation gains are tuned to be $Q_\alpha^{-1} = 5e-8$ and $Q_\beta^{-1} = 1e-17$, and the design parameter $\eta = 1$. The reference trajectory is defined by passing a reference signal through a linear reference model $Y_m(s)/R(s) = 3/(s+3)$ so that $y_m(t)$ and $\dot{y}_m(t)$ are measurable and bounded. Since the time-varying dynamics caused by both incipient faults and jump-like faults satisfy the assumptions on the bounded parameter variations and bounded parameter rate of change, we could apply the stable adaptive control method developed above to solve the fault-tolerant engine control problem.

We let the component level engine model run at the operating condition of ALT = 15 000, XM = 0.7, DTAMB = 0, and PC = 46. For engine quality parameters, we set the initial engine variation to be $p_{iev} = [0.1\%, 0.1\%, 0.2\%, 0.1\%, -0.1\%, 0, -0.3\%, 0.2\%, -0.1\%, 0.1\%]$, and the engine deterioration index to be $I_d = 0.1$. Figure 1 shows the control performance of indirect adaptive control for a multiple fault scenario, where an incipient-type fault evolves from no fault to a large fan fault over 20 s, and an abrupt large compressor hub fault occurs at $T_f = 6$ s. The occurrence of the abrupt large compressor hub fault affects the system performance drastically (as indicated by arrow 1), whereas after a period of time for learning the effects of the fault on system dynamics, the adaptive controller accommodates for

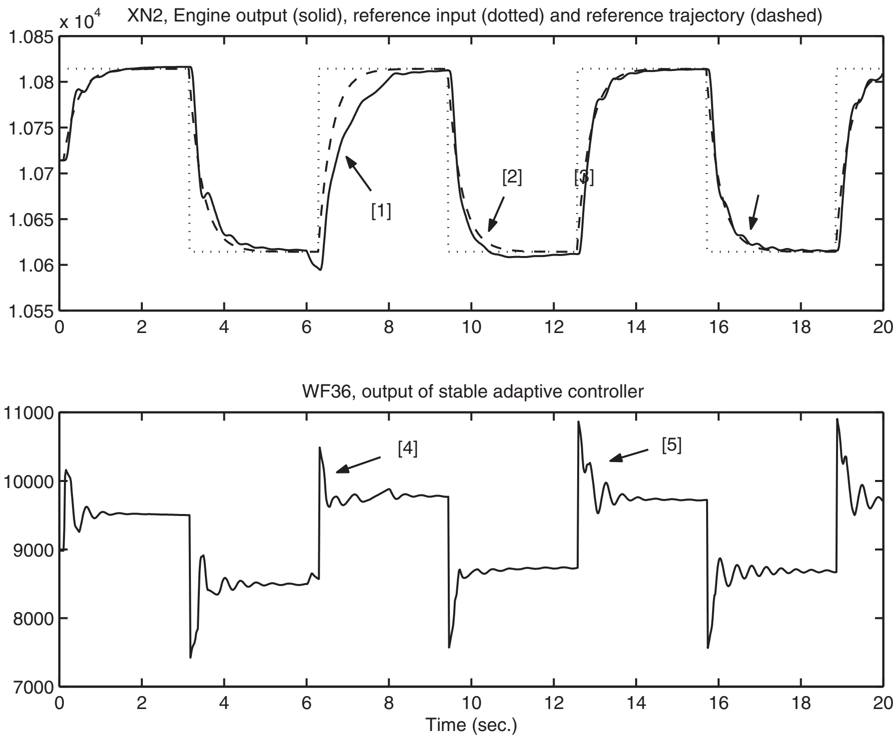

Figure 1. Performance of indirect adaptive controller.

the fault quite well (as indicated by arrows 2 and 3) by modifying the control action (as indicated by arrows 4 and 5). As for the incipient fan fault, since its effects are not so significant compared to the fault accommodation ability of the adaptation scheme, no apparent control performance deterioration can be seen.

The effectiveness of the proposed stable adaptive controller can be demonstrated through comparing its performance with that of a nominal controller. By removing on-line approximators (21) and (22) and turning off the sliding mode control term (37), the nominal controller is in a kind of state feedback form

$$u = \frac{1}{\beta_k(c, p_0, t)}(-\alpha_k(c, p_0, t) + \nu(t)) \tag{87}$$

where $\alpha_k(c, p_0, t)$ and $\beta_k(c, p_0, t)$ represent nominal model dynamics. The control performance of this nominal controller, as shown in figure 2, is much worse than that of the adaptive controller. Since no adaptation scheme is involved, the controller could generate good performance when the faults are not too significant (e.g., for the first 2 s in figure 2), but results in deteriorated performance when the faults become serious and thus the system dynamics leave far away from the nominal dynamics. The effectiveness of the adaptation scheme can be further clarified by noting that, as shown in figure 3, if we keep the on-line approximators but turn off the sliding mode control term, the adaptive controller is still able to achieve fault accommodation abilities even though the control performance is not as good as in figure 1. However, if we keep the sliding mode control term but remove the on-line approximators, as shown in figure 4, the faults are not accommodated for.

6.2. *Direct adaptive control*

We also apply the direct adaptive control scheme to the fault-tolerant engine control problem. For direct adaptive control scheme, the nominal engine model cannot be used. Instead, we define the known controller to be a proportional-integral (PI) controller ($u_k = k_p(e + 1/T_i \int e\, dt)$, $k_p = 5, T_i = 0.2$). By studying dynamics of the developed non-linear model, we know that $g_1(x, c, p) < 0.38$ and its rate of change is smaller than 1.5 so that we can set $\beta_1 = 0.38$ and $B = 1.5$. The ideal controller is approximated by one radial basis function network with 11 receptive field units. The inputs to the neural network include two state variables (XNL and XNH) as well as the variable ν, and the parameters are updated on-line to compensate for the unknown time-varying dynamics affected by model inaccuracy and faults. By trial and error, the model uncertainty is described by $W_u = 200$, the adaptation gain is $Q_u^{-1} = 2e - 7$ and the design parameter is chosen to be $\eta = 1$.

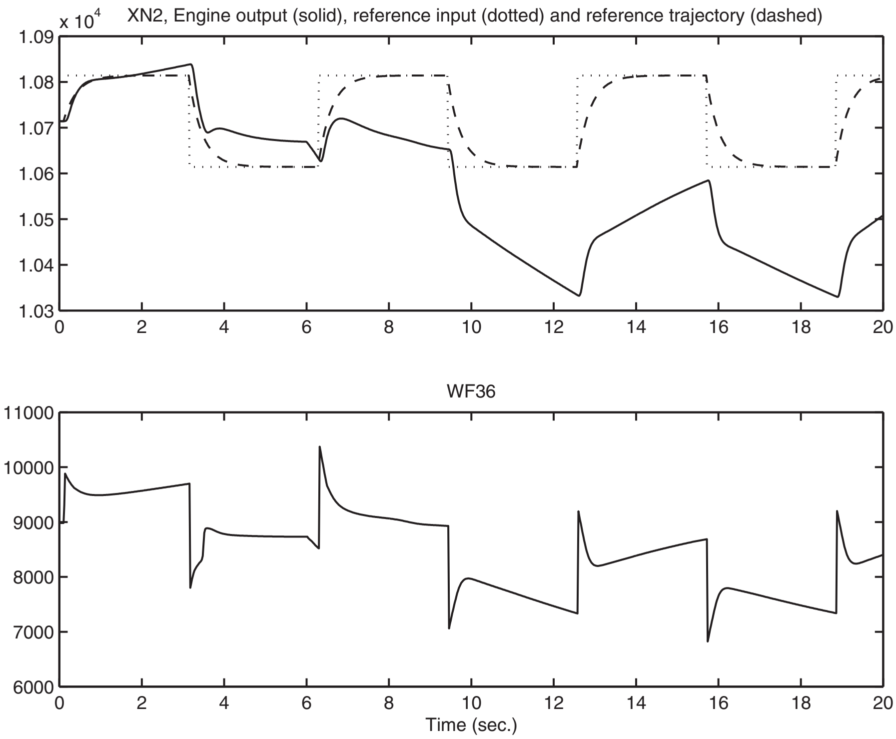

Figure 2. Performance of the state feedback controller.

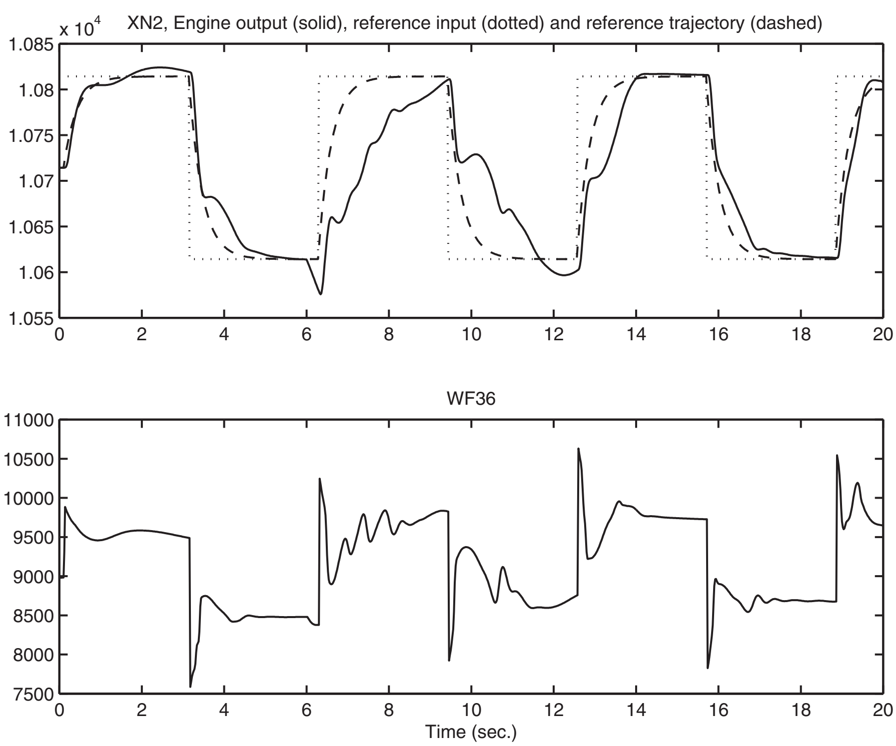

Figure 3. Control performance without the sliding mode control term.

Compare the results of the direct adaptive controller, as shown in figure 5, with those of the indirect adaptive controller (in figure 1) for the same fault scenario. Generally, even for other choices of design parameters, the direct adaptive controller seems to learn and adapt faster than the indirect adaptive controller (probably because it has fewer parameters to be tuned). However, note that there is more oscillation in the direct

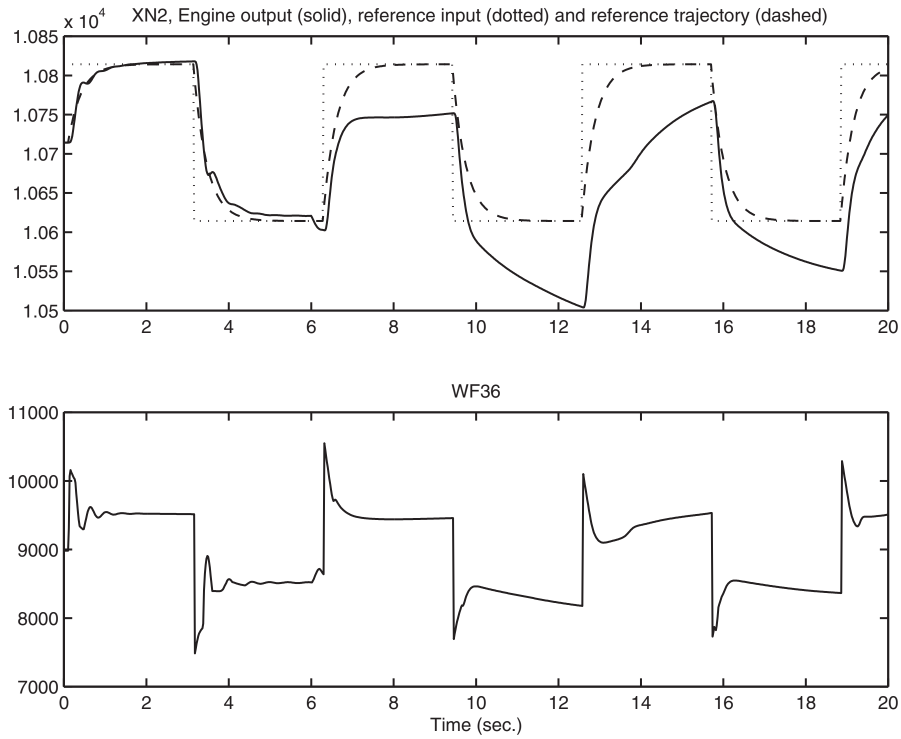

Figure 4. Control performance without on-line approximators.

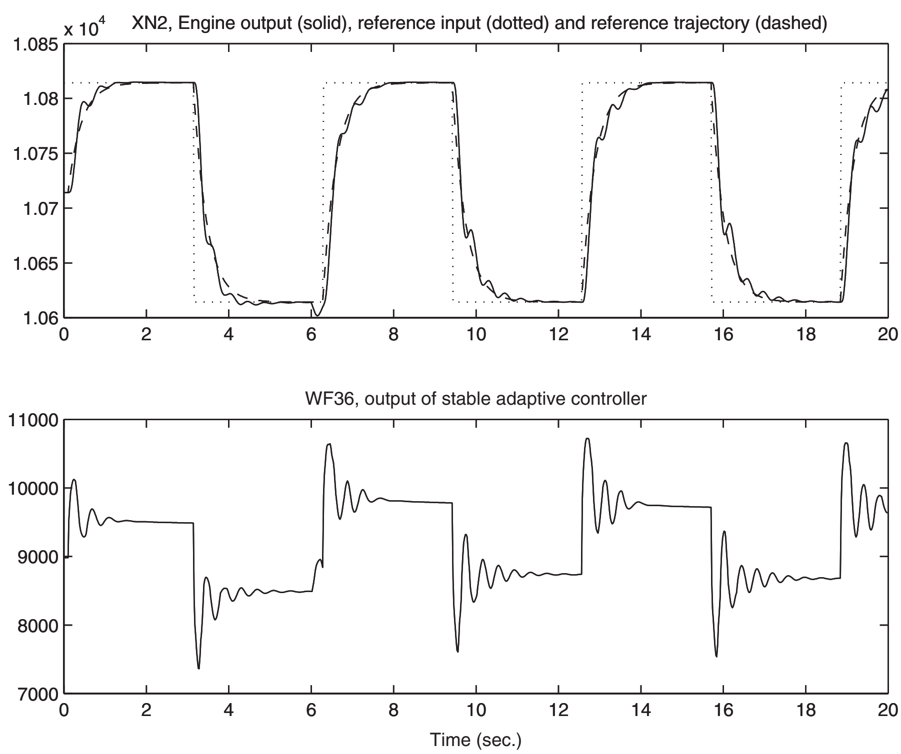

Figure 5. Performance of the direct adaptive controller.

case compared to the indirect case. This is because the direct adaptive controller cannot use a priori knowledge of the engine from the nominal engine model. Instead, it uses a known controller which is not good enough, so that the control action relies more heavily on the adaptation scheme.

7. Conclusions

Fault-tolerant system design for non-linear time-varying systems can be quite challenging. Most existing studies on fault diagnosis and fault-tolerant control have relied on a linear nominal model of the plant. In practical situations, however, plants are non-linear and the faults often force plants away from local linear behaviours into non-linear operating regions. Furthermore, the existing work in the literature mainly considers fault-tolerant control in the context of time-invariant systems as if a fault has already occurred, but the reality is that both incipient and abrupt faults are naturally time-varying phenomena. In this paper we have presented on-line approximation-based stable adaptive neural/fuzzy control methods for a class of input–output feedback linearizable time-varying non-linear systems. This class of systems is large enough so that it is not only of theoretical interest but also of practical applicability (e.g. to the fault tolerant control problem of the General Electric XTE46 engine that we encountered in a project funded by NASA). The adaptive control problem has been reformulated in the time-varying context and new adaptive control laws have been designed to generalize the existing robust adaptive fuzzy/neural control method to time-varying cases by taking into account uncertain time-varying parameters (with known bounds). Uniform boundedness of all internal signals and 'small in the mean' tracking of a reference signal have been obtained under the assumption of bounded time-varying parameters. Uniform asymptotic stability of the system output can be further achieved by assuming boundedness of parameter rate of change. Both indirect and direct adaptive control methods have been studied for this class of non-linear time-varying systems, while the direct method is, to the best of our knowledge, the first such kind of approaches of adaptive control for non-linear time-varying systems. The effectiveness of the adaptive control methods proposed in this paper has been demonstrated using the component level model simulation of the XTE46 engine. Unlike the typical engine models that are used in some of the literature, this XTE46 simulator has been developed by GEAE to be very complicated and accurate so that the simulation conducted on this simulator is very close to that on the real engine for actual flights.

There are several issues in the proposed research that could be further studied. One of them concerns our assumptions about having a bounded parameter rate of change in adaptive control in order to obtain the uniform asymptotic stability of the system output. These assumptions may be reasonable (e.g. the tracking error is usually large if the plant parameters vary fast). However, these conditions may be difficult to verify in specific applications. Thus, it is still an open problem to find necessary assumptions easy to be verified and still capable of guaranteeing uniform asymptotic stability.

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