A Separation Principle for Non-UCO Systems: The Jet Engine Stall and Surge Example

Manfredi Maggiore and Kevin M. Passino

***Abstract*—The problem of controlling surge and stall in jet engine compressors is of fundamental importance in preventing damage and lengthening the life of these components. In this theoretical study, we illustrate the application of a novel output feedback control technique to the Moore–Greitzer mathematical model for these two instabilities assuming that the plenum pressure rise is measurable. This problem is particularly challenging since the system is not *uniformly completely observable* and, hence, none of the output feedback control techniques found in the literature can be applied to recover the performance of a full state feedback controller.**

***Index Terms*—Nonlinear control, nonlinear observer, output feedback, separation principle, surge and stall.**

I. Introduction and Problem Description

We consider the problem of controlling two instabilities which occur in jet engine compressors, namely rotating stall and surge. In [8], Moore and Greitzer developed a three-state finite dimensional Galerkin approximation of a nonlinear PDE model describing the compression system. Since its development, several researchers have used the Moore–Greitzer three state model (MG3) to design stabilizing controllers for stall and surge; see, for instance, [3], [5], and [9]. Most existing results focus on the development of state feedback controllers which may not be implementable because the state is not entirely measurable. In [3], a partial state feedback controller simplifies practical implementation by only requiring measurements of the mass flow and plenum pressure rise.

To the best of our knowledge, available solutions to the output feedback control problem using only plenum pressure rise (see [1] and [2, Sec. 12.6, 12.7]) do not rely on the estimation of the entire state of the system, and it seems that no attempt has been made to design a stabilizing output feedback controller (using only plenum pressure rise feedback) based on a full-state feedback control law. In this note, we introduce a new globally stabilizing full state feedback control law for MG3, and we employ the theory developed in [7] for the output feedback control of non-UCO systems (i.e., system that are not globally observable) to regulate stall and surge by using only pressure measurements. We stress that the details of a practical design and implementation are not within the scope of this note.

The MG3 model is described by (see [3] for an analogous exposition)

$$
\begin{aligned}
\dot{\Phi} &= -\Psi + \Psi_C(\Phi) - 3\Phi R \\
\dot{\Psi} &= \frac{1}{\beta^2}(\Phi - \Phi_T) \\
\dot{R} &= \sigma R(1 - \Phi^2 - R),\ R(0) \geq 0
\end{aligned} \tag{1}
$$

Manuscript received March 21, 2002; revised July 23, 2002. Recommended by Associate Editor M. M. Polycarpou. This work was supported by the NASA Glenn Research Center under Grant NAG3-2084. Portions of this work were performed while M. Maggiore was a student at The Ohio State University, Columbus, OH 43210 USA.

M. Maggiore is with the Department of Electrical and Computer Engineering, University of Toronto, Toronto, ON M5S 3G4, Canada (e-mail: maggiore@control.utoronto.ca).

K. M. Passino is with the Department of Electrical Engineering, The Ohio State University, Columbus, OH 43210 USA (e-mail: k.passino@osu.edu).

Digital Object Identifier 10.1109/TAC.2003.814274

where Φ represents the mass flow, Ψ is the plenum pressure rise, $R \geq 0$ is the normalized stall cell squared amplitude, Φ_T is the mass flow through the throttle (throughout this note, we will set $\sigma = 7$, and $\beta = 1/\sqrt{2}$). The functions $\Psi_c(\Phi)$ and $\Phi_T(\Psi)$ are the compressor and throttle characteristics, respectively, and are defined as $\Psi_C(\Phi) = \Psi_{C_0} + 1 + 3/2\Phi - 1/2\Phi^3$, $\Psi = (1/\gamma^2)(1 + \Phi_T(\Psi))^2$, where Ψ_{C_0} is a constant and γ is the throttle opening, the control input. Our control objective is to stabilize system (1) around the critical equilibrium $R^e = 0$, $\Phi^e = 1$, $\Psi^e = \Psi_C(\Phi^e) = \Psi_{C_0} + 2$, which achieves the peak operation on the compressor characteristic. Shifting the origin to the desired equilibrium with the change of variables $\phi = \Phi - 1$, $\psi = \Psi - \Psi_{C_0} - 2$ we obtain

$$
\begin{aligned}
\dot{R} &= -\sigma R^2 - \sigma R(2\phi + \phi^2) \\
\dot{\phi} &= -\psi - \frac{3}{2\phi^2} - \frac{1}{2\phi^3} - 3R\phi - 3R \\
\dot{\psi} &= \frac{1}{\beta^2}\left(\phi - \gamma\sqrt{\psi + \Psi_{C_0} + 2} + 2\right)
\end{aligned} \tag{2}
$$

We assume the pressure rise (and hence ψ) to be the only measurable state variable.

II. State Feedback Control Design

We start by designing a full-state feedback controller which makes the origin of (2) an asymptotically stable equilibrium point with domain of attraction $\{(R, \phi, \psi) \in \mathbb{R}^3 | R \geq 0\}$, as seen in the next theorem.

Theorem 1: For (2), with the choice of the control law

$$
\bar{\gamma} = \frac{2 + (1 - \beta^2 k_1 k_2)\phi + \beta^2 k_2 \psi + 3\beta^2 k_1 R\phi}{\sqrt{\psi + \Psi_{C_0} + 2}} \tag{3}
$$

where k_1 and k_2 are positive scalars satisfying the inequalities

$$
\begin{aligned}
k_1 &> \frac{17}{8} + \frac{(2C\sigma + 3)^2}{2}\left(C\sigma - \frac{105}{64}\right)k_1^2 \\
&\quad + \frac{3}{4}\left(-\frac{1}{2}C\sigma + \frac{21}{4}\right)k_1 - (C\sigma + 3)^2 > 0 \\
k_2 &> k_1 + \frac{9}{4}k_1^2 + \frac{9k_1}{4k_1 - \frac{9}{2}} + \frac{\left(k_1^2 - 1\right)^2}{4}, \qquad C > \frac{3}{2\sigma}
\end{aligned} \tag{4}
$$

the origin is asymptotically stable with domain of attraction $\mathcal{A} = \{(R, \phi, \psi) \in \mathbb{R}^3 | R \geq 0\}$.

Proof: Without loss of generality, let $u = (1/\beta^2)(\phi - \gamma\sqrt{\psi + \Psi_{C_0} + 2} + 2)$, so that the last equation in (2) becomes $\dot{\psi} = u$. Next, notice that (2) can be viewed as the interconnection of two subsystems

$$
[S_1]\dot{R} = -\sigma R^2 \quad [S_2]\begin{cases} \dot{\phi} = -\psi - \frac{3}{2}\phi^2 - \frac{1}{2}\phi^3 \\ \dot{\psi} = u. \end{cases}
$$

Consider the following Lyapunov function candidate (partly inspired by [4, Sec. 2.4.3]) for (2), $V = CR + (1/2)\phi^2 + (k_1/8)\phi^4 + (1/2)(\psi - k_1\phi)^2$, where $C > 0$ is a scalar. After noticing that V is positive definite on the domain $\mathcal{A}$, and letting $\tilde{\psi} = \psi - k_1\phi$, we calculate the time derivative of V as follows:

$$
\begin{aligned}
\dot{V} = &-C\sigma R^2 - C\sigma R(2\phi + \phi^2) + \left(\phi + \frac{k_1}{2}\phi^3\right) \\
&\times\left(-\psi - \frac{3}{2}\phi^2 - \frac{1}{2}\phi^3 - 3R\phi - 3R\right) + \tilde{\psi} \\
&\times\left(u + k_1\psi + \frac{3}{2}k_1\phi^2 + \frac{1}{2}k_1\phi^3 + 3k_1R\phi + 3k_1R\right).
\end{aligned} \tag{5}
$$

Here, as in [4], we use the identity

$$-\frac{3}{2}\phi^2 - \frac{1}{2}\phi^3 = -\frac{1}{2}\left(\phi + \frac{3}{2}\right)^2 \phi + \frac{9}{8}\phi$$

to eliminate the potentially destabilizing term $-(\phi + k_1/2\phi^3)3/2\phi^2$. Next, substituting (3) into (5) (after taking in account the definitions of u and γ), letting $\bar{k}_1 = k_1 - 9/8$, and using the definition of $\tilde{\psi}$, we get

$$\begin{aligned}\dot{V} =& -C\sigma R^2 - C\sigma R(2\phi+\phi^2) + \left(\phi + \frac{k_1}{2}\phi^3\right) \\ &\times \left(-\tilde{\psi} - \bar{k}_1\phi - \frac{1}{2}\left(\phi + \frac{3}{2}\right)^2 \phi - 3R\phi - 3R\right) \\ &+ \tilde{\psi}\left(-(k_2 - k_1)\tilde{\psi} + k_1^2\phi + \frac{3}{2}k_1\phi^2 + \frac{1}{2}k_1\phi^3 + 3k_1 R\right) \\ \leq& -C\sigma R^2 - (2C\sigma+3)R\phi - (C\sigma+3)R\phi^2 - \bar{k}_1\phi^2 \\ &- \left(\frac{k_1\bar{k}_1}{2} + \frac{3k_1}{2}R\right)\phi^4 - \frac{3k_1}{2}R\phi^3 \\ &+ \tilde{\psi}\left(-(k_2 - k_1)\tilde{\psi} + \left(k_1^2 - 1\right)\phi + \frac{3}{2}k_1\phi^2 + 3k_1 R\right).\end{aligned}$$

By using Young's inequality[1] one can show that (refer to [6] for a detailed derivation)

$$\begin{aligned}\dot{V} \leq& -\begin{bmatrix} R \\ \phi^2 \end{bmatrix}^\top \begin{bmatrix} C\sigma - \frac{3}{2} & \frac{1}{2}\left(C\sigma + 3 - \frac{3}{8}k_1\right) \\ \frac{1}{2}\left(C\sigma + 3 - \frac{3}{8}k_1\right) & \frac{1}{4}k_1\bar{k}_1 \end{bmatrix} \\ &\times \begin{bmatrix} R \\ \phi^2 \end{bmatrix} - \left(\bar{k}_1 - \frac{(2C\sigma+3)^2}{2} - 1\right)\phi^2 \\ &- \left(k_2 - k_1 - \frac{9}{4}k_1^2 - \frac{9k_1}{4\bar{k}_1} - \frac{\left(k_1^2 - 1\right)^2}{4}\right)\tilde{\psi}^2.\end{aligned}$$

Using the inequalities in (4) we conclude that $\dot{V}$ is negative definite on the domain $\mathcal{A}$. This and the fact that the boundary of $\mathcal{A}$, $\partial\mathcal{A} = \{(R,\phi,\psi) | R = 0\}$, is an invariant manifold prove that the origin of the closed-loop system in an asymptotically stable equilibrium point and the set $\{(R,\phi,\psi) | V \leq K\} \cap \mathcal{A}$ is its region of attraction for any positive real number K. This, in turn, shows that $\mathcal{A}$ is the domain of attraction of the origin of the closed-loop system. ■

In practice, k_1 and k_2 can be chosen significantly smaller than their theoretical lower bounds in (4). Choosing $\beta = 7$ and $\sigma = 1/\sqrt{2}$, we found that the smallest values of k_1 and k_2 satisfying (4) are given by $k_1 = 20.43$, $k_2 = 4.43 \cdot 10^4 (C = 0.2179)$. However, simulations of the closed-loop system (not included here for space limitations; see [6]) for several different initial conditions indicate that k_1 and k_2 can be chosen as low as 10.

Generally a full-state feedback controller may yield a better closed-loop performance than one using partial-state feedback because it uses more information about the state of the system. When comparing our full-state feedback controller to the partial-state feedback controller developed in [3], however, this claim cannot be made without a rigorous analysis which is beyond the scope of this note.

III. Output Feedback Design

In this section, we apply the methodology developed in [7] to recover the performance of the state feedback controller (3) using output feedback. In what follows, we summarize the main result in [7]. Consider the following dynamical system:

$$\begin{aligned}\dot{x} &= f(x,u) \\ y &= h(x,u)\end{aligned} \tag{6}$$

where $x \in \mathbb{R}^n$, $u, y \in \mathbb{R}$, f, and h are known smooth functions, and $f(0,0) = 0$. We want to design a stabilizing controller for (6) without the availability of the system states x. In order to do so, we need an observability assumption. Define the observability mapping $\mathcal{H}$ by calculating $n-1$ derivatives of y along the vector field f

$$y_e \triangleq \left[y, \ldots, y^{(n-1)}\right]^\top = \mathcal{H}\left(x, u, \ldots, u^{(n_u - 1)}\right) \tag{7}$$

where n_u, $0 \leq n_u \leq n$, denotes the number of time derivatives of u that appear in $\mathcal{H}$ ($n_u = 0$ indicates that there is no dependence on u). Next, augment the system dynamics with n_u integrators at the input side, which corresponds to using a compensator of order n_u

$$\dot{x} = f(x, z_1), \dot{z}_1 = z_2, \ldots, \dot{z}_{n_u} = v \tag{8}$$

so that (7) can be written as $y_e = \mathcal{H}(x,z)$. Let $X = [x^\top, z^\top]^\top \in \mathbb{R}^{n+n_u}$ denote the state variable of the *extended system*. We are now ready to state our first assumption.

Assumption A1 (Observability): System (6) is observable over an open set $\mathcal{O} \subset \mathbb{R}^n \times \mathbb{R}^{n_u}$ containing the origin, i.e., the mapping $\mathcal{F} : \mathcal{O} \to \mathcal{Y}$ (where $\mathcal{Y} = \mathcal{F}(\mathcal{O})$) defined by

$$Y = \left[y_e^\top, z^\top\right]^\top = \mathcal{F}(X) = \left[\mathcal{H}(x,z)^\top, z^\top\right]^\top \tag{9}$$

has a smooth inverse $\mathcal{F}^{-1} : \mathcal{Y} \to \mathcal{O}$, $\mathcal{F}^{-1}(Y) = \mathcal{F}^{-1}(y_e, z) = [\mathcal{H}^{-1}(y_e,z)^\top, z^\top]^\top$.

Following the terminology in [10], when $\mathcal{O} = \mathbb{R}^{n+n_u}$ we say that the system is *uniformly completely observable (UCO)*.

Assumption A2 (Stabilizability): There exists a smooth function $\bar{u}(x)$ such that the origin of (6) is an asymptotically stable (or globally asymptotically stable) equilibrium point of $\dot{x} = f(x, \bar{u}(x))$.

Using A2, the knowledge of a Lyapunov function for (6) with $u = \bar{u}(x)$, and the integrator backstepping lemma (see, e.g., [4]), one may design a smooth control law $v = \phi(x,z) = \phi(X)$ which makes the origin of (8) an asymptotically stable equilibrium point. In particular, from the application of the integrator backstepping lemma one also gets a Lyapunov function $\bar{V}(X)$. Given any scalar $c > 0$, let Ω_c denote the generic level set of $\bar{V}$, i.e., $\Omega_c \triangleq \{X \in \mathbb{R}^{n+n_u} | \bar{V} \leq c\}$. Our last assumption concerns the topology of the "observability set" $\mathcal{O}$.

Assumption A3 (Topology of $\mathcal{O}$): Assume that there exists a constant $c_2 > 0$ and a set $\mathcal{C}$ such that $\mathcal{F}(\Omega_{c_2}) \subset \mathcal{C} \subset \mathcal{Y}(= \mathcal{F}(\mathcal{O}))$, where $\mathcal{C}$ has the following properties.

i) The boundary of $\mathcal{C}$, $\partial\mathcal{C}$, is class C^1, i.e., there exists a C^1 function $g : \mathcal{C} \to \mathbb{R}$ such that $\partial\mathcal{C} = \{Y \in \mathcal{C} | g(Y) = 0\}$, and $(\partial g/\partial Y)^\top \neq 0$ on $\partial\mathcal{C}$.

ii) Each slice $\mathcal{C}^{\bar{z}} = \{y_e \in \mathbb{R}^n | [y_e^\top, \bar{z}^\top]^\top \in \mathcal{C}\}$ is convex for all $\bar{z} \in \mathbb{R}^{n_u}$.

iii) Zero is a regular value of $g(\cdot, \bar{z})$ for each fixed $\bar{z} \in \mathbb{R}^{n_u}$, i.e., $[\partial g/\partial y_e(y_e, \bar{z})]^\top$ does not vanish anywhere on the boundary of each slice $\mathcal{C}^{\bar{z}}$.

iv) $\bigcup_{\bar{z} \in \mathbb{R}^{n_u}} \mathcal{C}^{\bar{z}}$ is compact.

Given a real-valued function $x \longmapsto a(x)$, $\mathbb{R}^n \to \mathbb{R}$, and a vector field a in $\mathbb{R}^n$, recall that the Lie derivative $L_a b$ is defined as $L_a b = (\partial b/\partial x) a(x)$. We are now ready to introduce the output feedback controller for the extended system (8), shown in (10) and (11) at the bottom of the next page, and the various parameters are defined in the following table:

$\hat{y}_e^P = \mathcal{H}(\hat{x}^P, z)$	$\hat{Y}^P = \mathcal{F}(\hat{x}^P, z) = \left[\hat{y}_e^{P\top}, z^\top\right]^\top$
$N_{y_e}(\hat{Y}^P) = \left[\frac{\partial g}{\partial \hat{y}_e^P}\right]^\top$	$\mathcal{E} = \text{diag}[\rho, \ldots, \rho^n], \rho > 0$
$L \in \mathbb{R}^n$ Hurwitz	$\Gamma = \rho^{2n}(\mathcal{SE})^{-1}(\mathcal{SE})^{-\top}$

[1]For any real numbers a and b, and any positive real k, one has that $ab \leq (a^2/4k) + kb^2$.

with $S = S^\top = P^{(1/2)}$, where P is the solution of the Lyapunov equation $P(A_c - LC_c) + (A_c - LC_c)^\top P = -I$ and (A_c, C_c) is the canonical observable pair with eigenvalues at zero.

The controller (11) has a certainty equivalence structure. The observer with state $\hat{x}^P$ incorporates a dynamic projection which constrains the estimate $\hat{x}^P$ to lie inside the set $\mathcal{H}^{-1}(\mathcal{C}) \subset \mathcal{O}$ and thus guarantees its well definiteness. This feature is particularly useful when $\mathcal{O}$ is not all of $\mathbb{R}^{n+n_u}$ (that is, when the system is not UCO) and other output feedback control approaches based on a separation principle such as [10] cannot be employed. In the next section, we will show that MG3 is not UCO and will use the methodology presented here to solve the output feedback stabilization problem.

The following result states that (10) and (11) guarantee closed-loop stability.

Theorem 2 ([7]): For the closed-loop system (8), (10), (11), satisfying assumptions A1, A2, and A3, for any $0 < c_1 < c_2$ there exists a scalar ρ^*, $0 < \rho^* \leq 1$, such that, for all $\rho \in (0, \rho^*]$, the set $\{(X, \hat{x}^P) \in \mathbb{R}^{2n+n_u} | X \in \Omega_{c_1}, (\hat{x}^P, z) \in \mathcal{F}^{-1}(\mathcal{C})\}$ is contained in the region of attraction of the origin $(X, \hat{x}^P) = (0, 0)$.

We are now ready to apply the result of Theorem 2 to MG3. To this end, we start by verifying that assumptions A1–A3 hold for (2).

Observability: We form the mapping $\mathcal{H}$ from the measurable output $y = \psi$

$$y_e = [y, \dot{y}, \ddot{y}]^\top = \mathcal{H}\left([R, \phi, \psi]^\top, \gamma, \dot{\gamma}\right) = \begin{bmatrix} \psi \\ \frac{1}{\beta^2}\left(\phi - \theta(\psi, \gamma)\right) \\ \frac{1}{\beta^2}\left(-\psi - \frac{3}{2\phi^2} - \frac{1}{2\phi^3} - 3R\phi - 3R - \dot{\theta}\right) \end{bmatrix} \quad (12)$$

where, for convenience, we denoted $\theta(\psi, \gamma) = \gamma\sqrt{\psi + \Psi_{C_0} + 2} - 2$ and $\dot{\theta} = (\partial\theta/\partial\psi)\dot{\psi} + (\partial\theta/\partial\gamma)\dot{\gamma}$. Recall that γ is the control input and note that both γ and $\dot{\gamma}$ appear in $\mathcal{H}$, thus $n_u = 2$. Next, we need to augment the system with $n_u = 2$ integrators at its input side. To simplify the integrator backstepping design, we employ a chain of two integrators with a *modified output*

$$\dot{z}_1 = z_2 \quad \dot{z}_2 = v \quad \gamma = \frac{z_1 + 2}{\sqrt{\psi + \Psi_{C_0} + 2}} \quad (13)$$

so that θ and $\dot{\theta}$ in (12) are replaced by z_1 and z_2, respectively, and the augmented system becomes the following cascade interconnection of two subsystems $[P_1]$ and $[P_2]$

$$[P_1]\begin{cases} \dot{R} = -\sigma R^2 - \sigma R(2\phi + \phi^2) \\ \dot{\phi} = -\psi - \frac{3}{2\phi^2} - \frac{1}{2\phi^3} - 3R\phi - 3R \\ \dot{\psi} = \frac{1}{\beta^2}(\phi - z_1) \end{cases}$$
$$[P_2]\begin{cases} \dot{z}_1 = z_2 \\ \dot{z}_2 = v. \end{cases} \quad (14)$$

Note that the dynamic extension (13) is well-defined in an output feedback setting because the output of (13) is a function of the measurable variables z_1 and ψ. Next, the mapping $\mathcal{F}$ is given by $Y = \mathcal{F}([R, \phi, \psi]^\top, [z_1, z_2]^\top) = [\mathcal{H}([R, \phi, \psi]^\top, z_1, z_2)^\top, z_1, z_2]^\top$. Notice that the observability assumption A1 is satisfied on the set $\mathcal{O} = \{[R, \phi, \psi]^\top \in \mathbb{R}^3, z \in \mathbb{R}^2 | \phi > -1\}$ and, hence, the system *is not UCO*. It is easy to check that, when $\phi = -1$ and, hence, $\Phi = 0$, $\mathcal{F}$ does not depend on R and, hence, it is not invertible. Hence, when there is no mass flow through the compressor $(\Phi = 0)$ the normalized stall cell squared amplitude R cannot be observed. Clearly, $\Phi = 0$ is a condition we would like to avoid during normal engine operation.

Stabilizability: To be consistent with the notation used earlier, let $x = [R, \phi, \psi]^\top$. Rewrite $[P_1]$ in (14) as $\dot{x} = f_1(x) + g_1(x)z_1$ (also, let $f(x, z_1) = f_1(x) + g_1(x)z_1$). From Theorem 1 we have that the stabilizability assumption A2 is satisfied by the controller $\bar{\gamma}(x)$. Next, recalling that $z_1 = \theta$, in order to design a stabilizing control law for the extended system (14) one can view $[P_1]$ as a subsystem with input θ and stabilizing controller $\bar{\theta} = \bar{\gamma}(x)\sqrt{\psi + \Psi_{C_0} + 2} - 2$ and apply integrator backstepping. Doing so, one obtains the stabilizing control law

$$v = \dot{\alpha} - \tilde{z}_1 - k_4\tilde{z}_2 \triangleq \phi(x, z) \quad (15)$$

where $\tilde{z}_1 = z_1 - \bar{\theta}(x)$, $\alpha(x, z_1) = -k_3\tilde{z}_1 - (\partial V/\partial x)g(x) + (\partial\bar{\theta}/\partial x)[f(x) + g(x)z_1]$, $\tilde{z}_2 = z_2 - \alpha(x, z_1)$, and k_3, k_4 are arbitrary positive constants. This completes the design of a stabilizing state feedback for the extended system (14). The Lyapunov function of the closed-loop extended system is $\bar{V} = V + (1/2)\tilde{z}_1^2 + (1/2)\tilde{z}_2^2$, where V is defined in the proof of Theorem 1. Following the same reasoning as in the proof of Theorem 1, we conclude that the origin of the extended system is asymptotically stable with domain of attraction $\mathcal{D} = \mathcal{A} \times \mathbb{R}^2$

$$\hat{f} = \begin{bmatrix} -\sigma(\hat{R}^P)^2 - \sigma\hat{R}^P\left(2\hat{\phi}^P + (\hat{\phi}^P)^2\right) \\ -\frac{\frac{l_1}{\rho} + \frac{\beta^2 l_2}{\rho^2}\left(3\hat{\phi}^P + 3\hat{R}^P + \frac{3}{2}(\hat{\phi}^P)^2\right) + \frac{\beta^2 l_3}{\rho^3}}{3(1+\hat{\phi}^P)}(\psi - \hat{\psi}^P) \\ -\hat{\psi}^P - \frac{3}{2}(\hat{\phi}^P)^2 - \frac{1}{2}(\hat{\phi}^P)^3 - 3\hat{R}^P\hat{\phi}^P \\ -3\hat{R}^P + \frac{\beta^2 l_2}{\rho^2}(\psi - \hat{\psi}^P) \\ -\frac{z_1 - \hat{\phi}^P}{\beta^2} + \frac{l_1}{\rho}(\psi - \hat{\psi}^P) \end{bmatrix}. \quad (16)$$

Topology of the observability set: Noting that $\mathcal{Y} = \mathcal{F}(\mathcal{O}) = \{y_e \in \mathbb{R}^3, z \in \mathbb{R}^2 | y_{e,2} > (1/\beta^2)(-1 - z_1)\}$, it is readily seen that the set

$$\mathcal{C} = \left\{Y \in \mathbb{R}^5 | y_{e,1} \in [a_1, b_1], y_{e,2} \in \left[\frac{a_2 - z_1}{\beta^2}, \frac{b_2 - z_1}{\beta^2}\right], y_{e,3} \in \left[\frac{-z_2 + a_3}{\beta^2}, \frac{-z_2 + b_3}{\beta^2}\right], z_1 \in [a_4, b_4], z_2 \in [a_5, b_5]\right\}$$

parameterized by the set of scalars $\{a_i, b_i \in \mathbb{R} | a_i < b_i, i = 1, \ldots, 5\}$, is contained in $\mathcal{Y}$ for all $a_2 > -1$. Furthermore, each slice $\mathcal{C}^{\bar{z}}$ obtained from $\mathcal{C}$ by holding z constant at $\bar{z}$ is convex (it is a parallelepiped in

$$\dot{\hat{x}}^P = \begin{cases} \left[\frac{\partial\mathcal{H}}{\partial\hat{x}^P}\right]^{-1}\left\{L_{\hat{F}}\mathcal{H} - \Gamma\frac{N_{ye}(\hat{Y}^P)L_{\hat{G}}g}{N_{ye}(\hat{Y}^P)^\top\Gamma N_{ye}(\hat{Y}^P)} - \frac{\partial\mathcal{H}}{\partial z}\dot{z}\right\}, & \text{if } L_{\hat{G}}g \geq 0 \text{ and } \hat{Y}^P \in \partial\mathcal{C} \\ \hat{f}(\hat{x}^P, z, y), & \text{otherwise} \end{cases} \quad (10)$$

$$v = \phi(\hat{x}^P, z) \quad (11)$$

where

$$\hat{f}(\hat{x}^P, z, y) = f(\hat{x}^P, z_1) + \left[\frac{\partial\mathcal{H}(\hat{x}^P, z)}{\partial\hat{x}^P}\right]^{-1}\mathcal{E}^{-1}L\left(y - h(\hat{x}^P, z_1)\right)$$

$$\hat{F} = \left[\hat{f}(\hat{x}^P, z, y)^\top, z^\top\right]^\top \quad \hat{G} = L_{\hat{F}}\mathcal{F}$$

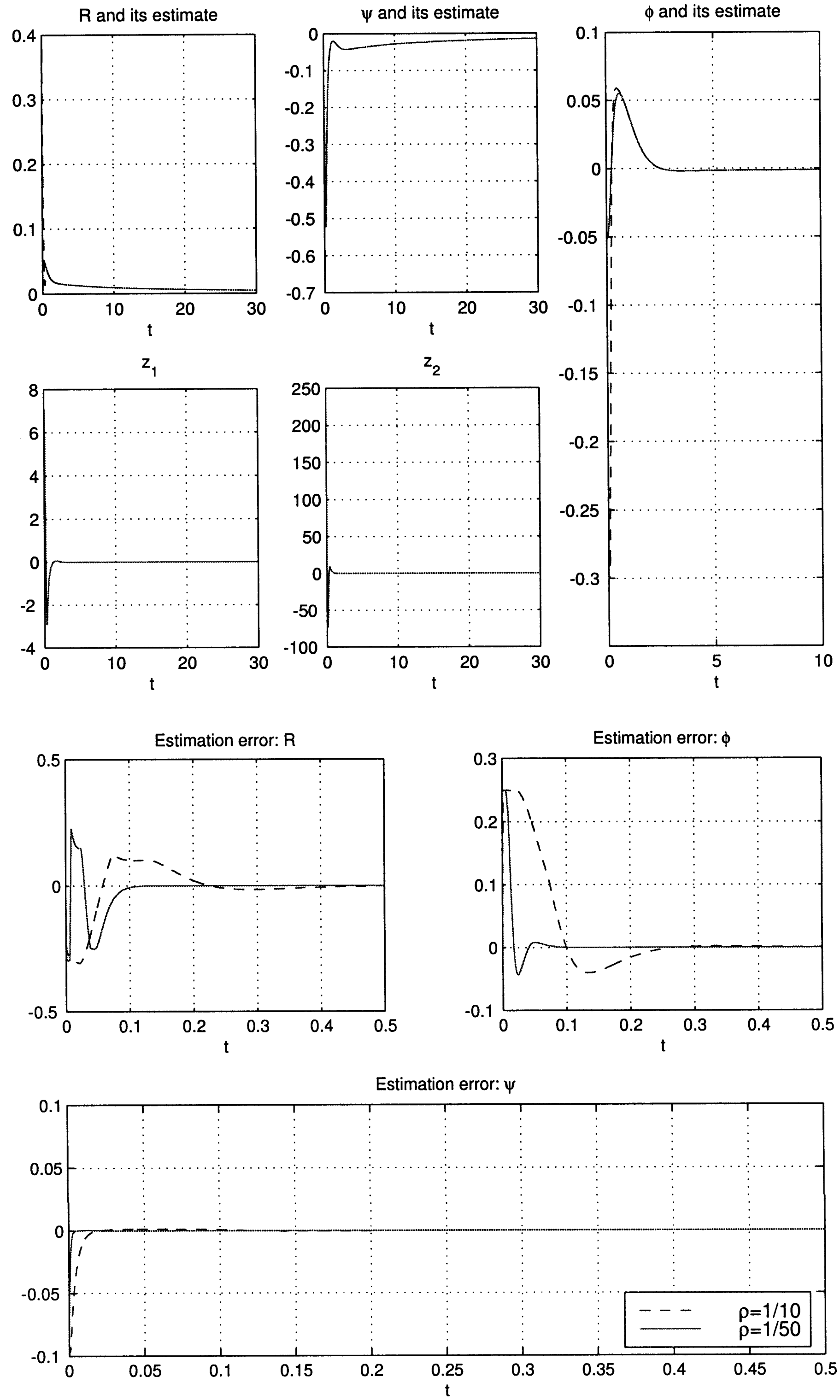

Fig. 1. Closed-loop system trajectories ($\rho = 1/5$) and estimation errors ($\rho = 1/10, 1/50$).

$\mathbb{R}^3$), thus satisfying requirement ii) in A3. The union of all slices $\mathcal{C}^{\bar{z}}$ is the set

$$\bigcup_{\bar{z}\in\mathbb{R}^2} \mathcal{C}^{\bar{z}} = \{y_e \in \mathbb{R}^3 | y_{e,1} \in [a_1, b_1],$$
$$y_{e,2} \in \left[\frac{a_2 - b_4}{\beta^2}, \frac{b_2 - a_4}{\beta^2}\right], y_{e,3} \in \left[\frac{-b_5 + a_3}{\beta^2}, \frac{-a_5 + b_3}{\beta^2}\right]\right\}$$

which is clearly compact, thus satisfying requirement (iv). Notice that the boundary of the set $\mathcal{C}$ defined above does not fully satisfy requirement (i) because it is continuous but not differentiable at some corners. This, in general, may generate some numerical problems in the projection which can be dealt with by smoothing out the corners of $\mathcal{C}$. Using the definition of $\mathcal{C}$ above one can calculate the vectors N_{y_e} and N_z (because of space limitations we omit their expression, see [6]) and verify that N_{y_e} never vanishes. So, in particular, N_{y_e} does not

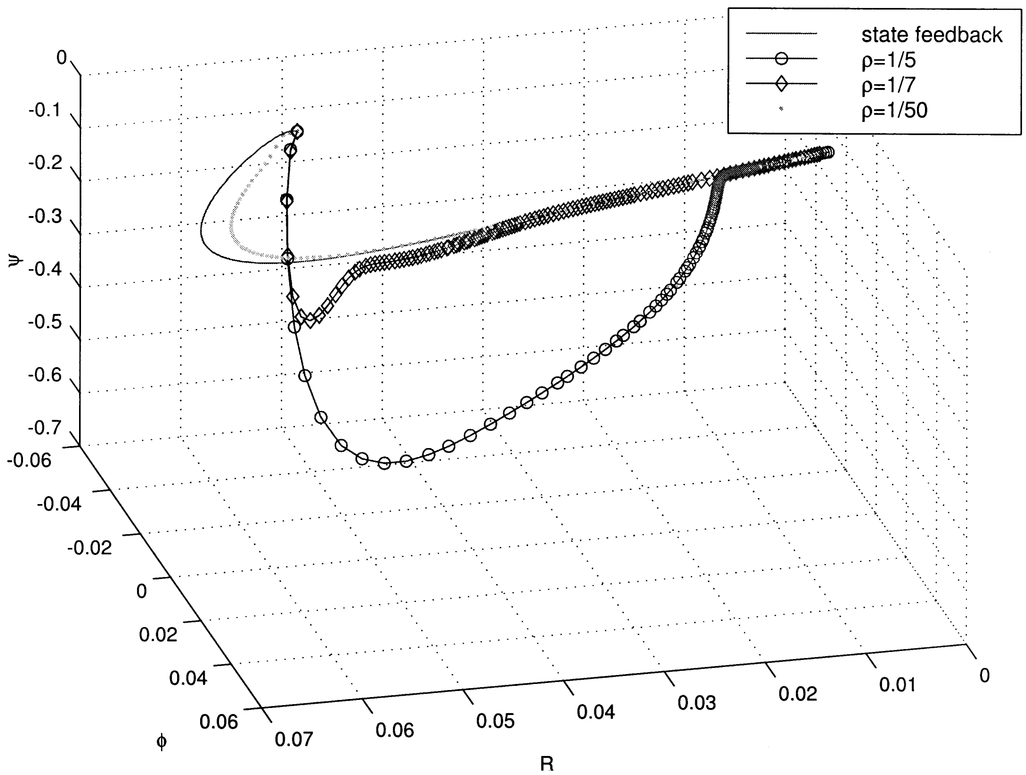

Fig. 2. State feedback trajectories and output feedback trajectories for several choices of ρ.

vanish on any slice $\mathcal{C}^{\bar{z}}$, and thus requirement iii) is fulfilled. In conclusion, in order for A3 to be satisfied, it remains to use the Lyapunov function $\bar{V}$ to find the largest value of c_2 such that $\Omega_{c_2} \subset \mathcal{O}$ (implying that $\mathcal{F}(\Omega_{c_2}) \subset \mathcal{F}(\mathcal{O})$) and, subsequently, pick values for the scalars $a_i, b_i, i = 1, \ldots, 5$ such that $a_2 > -1$ and $\mathcal{F}(\Omega_{c_2}) \subset \mathcal{C}$. A more practical way to address the design of $\{a_i, b_i\}$ entails running a number of simulations for the closed-loop system under state feedback corresponding to several initial conditions $(x(0), z(0))$ and calculating upper and lower bounds for $\psi(t), \phi(t), -\psi(t) - 3/2\phi^2(t) - 1/2\phi^3(t) - 3R(t)\phi(t) - 3R(t), z_1(t), z_2(t)$. By doing that, we found that whenever $[x(0)^\top, z(0)^\top]^\top \in \Omega_0 \triangleq \{[x(0)^\top, z(0)^\top]^\top \in \mathbb{R}^5 : R \in [0, 0.1], \phi \in [-0.1, 0.1], \psi \in [-0.5, 0.5], z_1 \in [-0.1, 0.1], z_2 \in [-0.1, 0.1]\}$, we have that $a_1 = -1.15$, $b_1 = 0.5$, $a_2 = -0.3$, $b_2 = -0.1$, $a_3 = -0.75$, $b_3 = 0.4$, $a_4 = -2$, $b_4 = 7$, $a_5 = -70$, and $b_5 = 250$. We must point out that our choice of Ω_0 is rather conservative and is made primarily for the sake of illustration.

Observer design: Having verified that assumptions A1–A3 hold and having selected the set $\mathcal{C}$, we are ready to design observer (10) for MG3. Denoting by $\hat{x}^P$ the vector $[\hat{R}^P, \hat{\phi}^P, \hat{\psi}^P]^\top$, the vector field $\hat{f}(\hat{x}^P, z, y)$ is given in (16). In conclusion, the output feedback controller design is given by $\hat{v} = \phi(\hat{x}^P, z)$, where the function ϕ is defined in (15).

IV. Simulation Results

Here, we present the simulation results when the output feedback controller developed in the previous section is applied to (2). We choose $k_1 = 20.43, k_2 = 4.43 \cdot 10^4$ to fulfill inequalities (4) in Theorem 1, and $L = [6, 12, 8]^\top$ so that the associated polynomial $s^3 + l_1 s^2 + l_2 s + l_3 = 0$ is Hurwitz. In Fig. 1 system and controller states, together with the control input, are plotted for $\rho = 1/5$. The figure clearly shows the operation of the projection which prevents the observer from peaking and guarantees that $\hat{\phi} > -0.3$ and, thus, is bounded away from the singularity in -1. Fig. 1 also depicts the evolution of the observer estimation error for $\rho = 1/10$ and $\rho = 1/50$, confirming the theoretical predictions of [7, Th. 1 and Lemma 1] concerning the arbitrary fast rate of convergence of the observer with projection (10). Finally, in Fig. 2 the orbits of (R, ϕ, ψ) are plotted for decreasing values of ρ.

V. Concluding Remarks

While existing separation principle approaches such as [10] cannot be applied to recover the performance of full-state feedback controllers for MG3, they can be employed to recover the performance of *any* partial state feedback controller which does not use R (such as the one in [3]), since the (ϕ, ψ) subsystem *is* UCO (whereas, as shown in earlier, R *is not* observable when $\phi = -1$). Additionally, without resorting to a separation principle, one can employ the technique developed in [2, Sec. 12.6, 12.7], and obtain semiglobal stabilization of the origin of the closed-loop system system, or the one presented in [1], based on a globally convergent observer and a small-gain design.

The modularity of our approach and, specifically, the availability of an estimate for the *full state* of the system provides some design flexibility in that it allows using available state feedback control design techniques. On the other hand, the results presented here have some limitations that need to be addressed. First, our methodology (as well as the approach in [10]) requires adding two integrators at the input side of MG3, thus unnecessarily complicating the state feedback design. Additionally, assuming, as we do, perfect knowledge of the compressor characteristic[2] and absence of disturbances is not a realistic assumption. We are currently working on extending our results in this direction.

References

[1] M. Arcak and P. Kokotović, "Nonlinear observers: a circle criterion design and robustness analysis," *Automatica*, no. 37, pp. 1923–1930, 2001.

[2]Note that [1]–[3] make the same assumption.

[2] A. Isidori, *Nonlinear Control Systems II*. London, U.K.: Springer-Verlag, 1999.
[3] M. Krstić, D. Fontaine, P. V. Kokotovic, and J. D. Paduano, "Useful nonlinearities and global stabilization of bifurcations in a model of jet engine surge and stall," *IEEE Trans. Automat. Contr.*, vol. 43, pp. 1739–1745, Dec. 1998.
[4] M. Krstić, I. Kanellakopoulos, and P. Kokotović, *Nonlinear and Adaptive Control Design*. New York: Wiley, 1995.
[5] D. Liaw and E. Abed, "Stability analysis and control of rotating stall," in *IFAC Nonlinear Control Systems Design Symp.*, Bordeaux, France, June 1992.
[6] Output feedback control of non-UCO systems: The jet engine stall and surge example, M. Maggiore and K. M. Passino. (2002). *http://www.control.utoronto.ca/tech_reports/rept0202.pdf* [Online]
[7] ——, "A separation principle for a class of non-UCO systems," *IEEE Trans. Automat. Contr.*, vol. 48, pp. 1122–1133, July 2003.
[8] F. K. Moore and E. M. Greitzer, "A theory of post-stall transients in axial compression systems-part I: development of equations," *J. Turbomach.*, vol. 108, pp. 68–76, 1986.
[9] J. D. Paduano, L. Valavani, A. Epstein, E. Greitzer, and G. R. Guenette, "Modeling for control of rotating stall," *Automatica*, vol. 30, no. 9, pp. 1357–1373, Sept. 1994.
[10] A. Teel and L. Praly, "Global stabilizability and observability imply semi-global stabilizability by output feedback," *Syst. Control Lett.*, vol. 22, pp. 313–325, 1994.

Contractibility of Dynamic LTI Controllers Using Complementary Matrices

Lubomír Bakule, José Rodellar, and Josep M. Rossell

***Abstract*—A generalized structure of complementary matrices involved in the input-state-output inclusion principle for linear time-invariant systems including contractibility conditions for static state feedback controllers is well known. In this note, it is shown how to further extend this structure when considering contractibility of dynamic controllers. Necessary and sufficient conditions for contractibility are proved in terms of both unstructured and block structured complementary matrices for general expansion/contraction transformation matrices. Explicit sufficient conditions for blocks of complementary matrices ensuring contractibility are proved for general expansion/contraction transformation matrices. Moreover, these conditions are further specialized for a particular class of transformation matrices.**

***Index Terms*—Contractibility, decentralized control, dynamic controllers, estimators, inclusion principle, overlapping.**

I. Introduction

The *inclusion principle* proposed in the context of analysis and control of complex and large scale systems in [10], [13], [14], and [16] establishes essentially a mathematical framework for two dynamic systems with different dimensions, in which solutions of the system with larger dimension include solutions of the system with smaller dimension. The relation between both systems is constructed usually on the base of appropriate linear transformations between the corresponding systems in the original and expanded spaces, where a key role in the selection of appropriate structure of all matrices in the expanded space is played by the so-called *complementary matrices* [11], [16]. The standard forms of complementary matrices such as aggregations and restrictions have been used in fact as the only well known forms for many years. A contribution to this issue has been presented in [1]–[5] giving a new procedure for a flexible selection of complementary matrices based on appropriate changes of basis in the systems.

When considering control, the following problem arises: give conditions to ensure that a controller designed for the expanded system can be transformed to be implemented in the original system in such a way that the inclusion principle holds for the closed-loop systems. A typical case in the literature is when an original system $\mathbf{S}$ with overlapped components is expanded to a bigger one with a number of disjoint subsystems. Then, decentralized controllers are designed in the expanded system $\tilde{\mathbf{S}}$ and then contracted for implementation in the original system $\mathbf{S}$. This scheme leads to the concept of contractibility.

Early work on contractibility was done for static state controllers in [9], [10], and [14], and for dynamic controllers (including estimators), in [6] and [12], but only with the use of standard complementary matrices in the context of aggregations and restrictions. Contractibility conditions of dynamic controllers were also derived in [7] and [8] for the particular expansion/contraction process referred to as extension, without using complementary matrices. Recently, contractibility of dynamic controllers has been revisited in a more general framework, in which a broader definition of contractibility is proposed to include the specific cases of restrictions, aggregations, and extensions [15], [17].

Suppose given dynamic controllers $\mathbf{C}$, $\tilde{\mathbf{C}}$ for $\mathbf{S}$, $\tilde{\mathbf{S}}$. [15, Th. 4] gives contractibility conditions in terms of the parameters of the closed-loop systems $(\mathbf{S},\ \mathbf{C})$ and $(\tilde{\mathbf{S}},\ \tilde{\mathbf{C}})$ without using complementary matrices. These conditions are general and have a fundamental character, since they do not assume any restriction on the systems and the controllers other than they are linear time-invariant (LTI). However, the conditions involve products and powers of the matrices defining $(\mathbf{S},\ \mathbf{C})$ and $(\tilde{\mathbf{S}},\ \tilde{\mathbf{C}})$. Therefore, there is not a direct way to derive the matrices of $(\tilde{\mathbf{S}},\ \tilde{\mathbf{C}})$ for given matrices of $(\mathbf{S},\ \mathbf{C})$ or viceversa. This makes the conditions difficult to be directly applied to set up expansion/contraction schemes in practical problems like, for instance, in control design.

This note relies on [15] to give structural properties of contractibility of dynamic controllers in expansion/contraction processes by using complementary matrices. The contractibility condition (D_2) in [15, Th. 4] is adopted here as the most important case for control design. It is restated in terms of complementary matrices without explicitly involving the matrices defining $(\mathbf{S}, \mathbf{C})$ and $(\tilde{\mathbf{S}}, \tilde{\mathbf{C}})$. This contractibility form results in explicit block structures of complementary matrices. These structures may potentially offer feasible degrees of freedom for specific choices of system matrices for building expansion/contractions schemes for specific problems. A previous work [1] has illustrated this potential in designing overlapping static linear quadratic optimal controllers.

II. Problem Statement

A. Preliminaries

Consider the LTI systems

$$\mathbf{S}: \dot{x} = Ax + Bu \qquad \tilde{\mathbf{S}}: \dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}\tilde{u}$$
$$y = Cx \qquad \tilde{y} = \tilde{C}\tilde{x} \tag{1}$$

Manuscript received July 30, 2002; revised February 24, 2003. Recommended by Associate Editor A. Garulli. This work was supported in part by the Generalitat of Catalonia under Grant DURSI PIV, by the Academy of Sciences of the Czech Republic under Grant A2075802, and by the Committee for Science and Technology (CICYT) of Spain under Grant DPI2002-04018-C02-01.

L. Bakule is with the Institute of Information Theory and Automation, Academy of Sciences of the Czech Republic, 182 08 Prague 8, Czech Republic (bakule@utia.cas.cz).

J. Rodellar and J. M. Rossell are with the Department of Applied Mathematics III, Universitat Politècnica de Catalunya (UPC), 08034 Barcelona, Spain (e-mail: jose.rodellar@upc.es; josep.maria.rossell@upc.es).

Digital Object Identifier 10.1109/TAC.2003.814275